Potential reachability in commutative nets

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Abstract. Potential reachability is a question about the linear structure of Petri nets. We prove a criterion for the solvability of the state equation in the case of commutative coloured nets. The proof relies on methods from commutative algebra and number theory. It generalizes the well-known criterion for potential reachability over $\boldsymbol{Z}$ for p/t nets .

Key words. Commutative net, potential reachability, Artin algebra, Dedekind domain.

Introduction

The linear theory of Petri nets is governed by the state equation. The state equation is an inhomogeneous linear equation. It computes the transition from an initial state to a final state due to the firing of one or more net transitions. Solvability of the state equation is therefore a necessary condition for the reachability of a given marking (*potential reachability*).

What this linear problem makes non-trivial, is the choice of the domain of coefficients: The strongest version considers the monoid $\boldsymbol{N}$ of non-negative integers, while relaxations allow coefficients from the ring $\boldsymbol{Z}$ or the field $\boldsymbol{Q}$. The problem acquires a new flavour in the realm of coloured Petri nets: In the present paper we study potential reachability for the subclass of commutative nets, the ring of coefficients will be the colour algebra $A_{\boldsymbol{Z}}$ of the net resp. its rational relaxation $A_{\boldsymbol{Q}}$. We prove:

- The state equation is solvable over the rational colour algebra $A_{\boldsymbol{Q}}$, iff it is solvable over the finitely many Artin factors of $A_{\boldsymbol{Q}}$. In the reduced case all Artin factors are number fields (Theorem 2.8).
- For a reduced net the state equation is not solvable over $A_{\boldsymbol{Z}}$, if is not solvable over the finitely many factors of the normalization of $A_{\boldsymbol{Z}}$. Each factor is a Dedekind domain (Theorem 3.3).
- Necessary and sufficient for the solvability of the state equation over a Dedekind domain is the equality concerning rank and Fitting ideal of the incidence matrix and the extended matrix (Theorem 3.4).

This paper generalizes previous results about reachability for p/t nets over $\boldsymbol{Z}$ ([DNM1996], [SW1999]). It continues our application of commutative algebra to Petri net theory.

1 Potential reachability over *Z* in p/t nets

Consider a finite p/t net N = (T, P, w^-, w^+) with transitions T, places P and weight functions

$$w^-, w^+: T \times P \rightarrow \boldsymbol{N}.$$

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We introduce $C_T(N)$ the free $\boldsymbol{Z}$-module with base T, its dual $C^T(N, \boldsymbol{Z}) := Hom_{\boldsymbol{Z}}(C_T(N), \boldsymbol{Z})$ as well as the tensor product $C_T(N, F) := C_T(N) \otimes_{\boldsymbol{Z}} F$ with an arbitrary $\boldsymbol{Z}$-module F. Similar expressions are used with P instead of T. We consider Parikh vectors as elements from $C_T(N)$ and markings as functionals from $C^P(N, \boldsymbol{Z})$. The incidence map is the $\boldsymbol{Z}$-linear map

$$w_T: C_T(N) \rightarrow C^P(N, \boldsymbol{Z}),\ [\ w_T\ (\ \Sigma_{t\in T}\ n_t\ t\ )\ ]\ (\ \Sigma_{p\in P}\ m_p\ p\ ) := \Sigma_{(t,p)\in TxP}\ n_t\ m_p\ [\ w^+(t, p) - w^-(t, p)\ ].$$

1.1 **Definition** *(Potential reachability)*

A marking M_{post} is *potentially reachable* over $\boldsymbol{Z}$ in the Petri net (N, M_{pre}), iff there exists a Parikh vector $\tau \in C_T(N)$, which satisfies the *state equation*

$$M_{post} = M_{pre} + w_T\ (\ \tau\ ).$$

1.2 **Theorem** *(Potential reachability over Z)*, ([SW1999], Theor. 6.4)

A marking M_{post} is potentially reachable over $\boldsymbol{Z}$ in the Petri net (N, M_{pre}), iff the incidence matrix and the extended incidence matrix have the same rank and the same ideal of minors (*Fitting ideal*), i.e. iff

- rank w_T = rank (w_T, Δ M) =: r
- and $< minor\ [\ r, w_T\ ] > = < minor\ [\ r, (\ w_T, \Delta M\ )\ ] > \subset \boldsymbol{Z}$.

Here $\Delta M := M_{post} - M_{pre}$ and $< minor\ [\ r, f\ ] >$ denotes the ideal generated by all minors of rank r of a linear endomorphism f.

2 Potential reachability over A_Q in commutative nets

To fix the notation we recall some definitions from [SW1999]. We denote by $X_{\boldsymbol{N}}$ the free monoid with basis a set X. For a ring A we denote by Spec A the set of its prime ideals and by Specm A the subset of maximal ideals. A general reference is [AM1969], [Bou1972].

2.1 **Definition** *(Homogeneous net)*

A *homogeneous net* N = (T, P, C, w^-, w^+) is a tuple with two disjoint finite sets T (*transitions*) and P (*places*), a finite set C (*colours*) and two families of *colour functions*

$$w^- = (w^-(t, p))_{(t,p)\in TxP},\ w^+ = (w^+(t, p))_{(t,p)\in TxP},\ \text{with } w^-(t, p), w^+(t, p) \in End_{\boldsymbol{N}}(C_{\boldsymbol{N}}).$$

2.2 **Definition** *(Colour algebra, commutative net)*

Consider a homogeneous net N = (T, P, C, w^-, w^+). The associative $\boldsymbol{Z}$-algebra generated by all colour functions

$$A_{\boldsymbol{Z}} := \boldsymbol{Z}\ [\ w^-, w^+\ ] \subset End_{\boldsymbol{Z}}\ (C_{\boldsymbol{Z}})$$

is called the *colour algebra* of N. The net N is *commutative*, iff $A_{\boldsymbol{Z}}$ is commutative.

2.3 **Proposition** *(Integrality of the colour algebra)*

The colour algebra of a commutative net is an integral extension of the ring $\boldsymbol{Z}$. It has an affine representation

$$A_{\boldsymbol{Z}} = \boldsymbol{Z}\,[t_1,...,t_k] \,/\, < h_1,...,h_p >.$$

Proof. Every endomorphism $f \in End_{\boldsymbol{Z}}(C_{\boldsymbol{Z}})$ is annihilated by its minimal polynomial, a uniquely determined normed polynomial

$$P_f(t) \in \boldsymbol{Z}\,[\,t\,]$$

with integer coefficients. Therefore every element from $A_{\boldsymbol{Z}}$ satisfies an integral equation and $A_{\boldsymbol{Z}}$ is an integral extension of $\boldsymbol{Z}$ ([AM1969], Chap. 5). Gauss' theorem implies, that the quotient representation holds also over $\boldsymbol{Z}$

$$\boldsymbol{Z}\,[\,f\,] \cong \boldsymbol{Z}\,[\,t\,] \,/\, < P_f(t) >.$$

We apply this consideration to the finitely many generators of $A_{\boldsymbol{Z}}$ and obtain a representation

$$A_{\boldsymbol{Z}} = \boldsymbol{Z}\,[t_1,...,t_k] \,/\, < h_1,...,h_p >$$

with polynomials $h_j \in \boldsymbol{Z}\,[t_1,...,t_k]$, j= 1,...,p, QED.

2.4 Proposition (Incidence map)

Consider a commutative net $N = (T, P, C, w^-, w^+)$ with colour algebra $A_{\boldsymbol{Z}}$.

i) The *incidence map at the level of the colour algebra* is a morphism between $A_{\boldsymbol{Z}}$-modules

$$w_{T,A\boldsymbol{Z}}: C_T(N, A_{\boldsymbol{Z}}) \rightarrow C^P(N, A_{\boldsymbol{Z}}),\; w_{T,A\boldsymbol{Z}}(t \otimes 1) := w(\,t, -\,) := w^+(\,t, -\,) - w^-(\,t, -\,).$$

ii) The evaluation of endomorphisms $A_{\boldsymbol{Z}} \subset End_Z(C_Z)$ provides an additional *$A_{\boldsymbol{Z}}$-module structure* for the colour module C_Z:

$$A_{\boldsymbol{Z}} \times C_Z \rightarrow C_Z,\; (a, c) \mapsto a(c).$$

iii) The *incidence map at the level of the colour module* is a morphism of $A_{\boldsymbol{Z}}$-modules

$$w_{T,C}: C_T(N, C_Z) \rightarrow C^P(N, C_Z),\; w_{T,C}(t \otimes c) := w(\,t, -\,)\,(c).$$

It derives from the incidence map at the level of the colour algebra as the tensor product

$$w_{T,C} = w_{T,A} \otimes_{A\boldsymbol{Z}} id_{CZ}.$$

2.5 Definition (Potential reachability, state equation)

Consider a commutative net $N = (T, P, C, w^-, w^+)$ with incidence map

$$w_{T,C}: C_T(N, C_Z) \rightarrow C^P(N, C_Z).$$

A marking M_{post} is *potentially reachable* over $A_{\boldsymbol{Z}}$ in the Petri net (N, M_{pre}), iff there exists a Parikh vector $\tau \in C_T(N, C_Z)$, which satisfies the *state equation*

$$M_{post} = M_{pre} + w_{T,C}(\,\tau\,).$$

2.6 Remark (The category $A_{\boldsymbol{Z}}$-Mod)

i) Proposition 2.4 gives evidence to our claim, that the colour algebra is the key algebraic object of a commutative net - not the colour module: The common incidence map derives from the incidence map at the level of the colour algebra. We will therefore concentrate on the category of A_Z-modules A_Z-Mod. We consider this category to be the right environment of high-level nets, not the "low-level" category of **Z**-modules.

ii) Torsion-free **$\boldsymbol{Z}$**-modules like $C_{\boldsymbol{Z}}$ are always free. But in general $C_{\boldsymbol{Z}}$ is not free as A_Z-module. Therefore we treat the question of reachability in two steps: First we compute the image of the incidence map at the level of the colour algebra. Then we determine the structure of the colour module considered as module over the colour algebra. Both steps can be easily handled in the reduced case over the rational colour algebra $A_{\boldsymbol{Q}}$.

2.7 Theorem *(Splitting over the rational colour algebra)*

Consider a commutative net N with colour algebra $A_{\boldsymbol{Z}}$ and colour module $C_{\boldsymbol{Z}}$.

i) The rational colour algebra $A_{\boldsymbol{Q}} := A_{\boldsymbol{Z}} \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ splits into a finite product of local Artin algebras

$$A_{\boldsymbol{Q}} \cong \Pi_{m\in \text{Specm } A\boldsymbol{Q}}\, A_m.$$

If $A_{\boldsymbol{Z}}$ is reduced, then every factor A_m is a number field.

ii) The rational colour module $C_{\boldsymbol{Q}} := C_{\boldsymbol{Z}} \otimes_{A\boldsymbol{Z}} A_{\boldsymbol{Q}}$ splits into a finite product of finitely generated A_m-modules

$$C_{\boldsymbol{Q}} \cong \Pi_{m\in \text{Specm } A\boldsymbol{Q}}\, C_m.$$

If $A_{\boldsymbol{Z}}$ is reduced, then every factor C_m is a finite dimensional vector space over A_m.

Proof. ad i) Proposition 2.3 implies, that the rational colour algebra $A_{\boldsymbol{Q}}$ is a finite-dimensional **$\boldsymbol{Q}$**-vector space, hence an Artin algebra. Every Artin algebra factors into a finite product of local Artin algebras ([AM1969], Chap. 8, Theor. 8.7). In the reduced case every factor A_m is reduced, too. A reduced local Artin algebra is a finite extension of the base field.

ad ii) The tensor product commutes with finite products, hence $C_m = C_{\boldsymbol{Q}} \otimes_{A\boldsymbol{Q}} A_m$, QED.

2.8 Theorem *(Potential reachability over $A_{\boldsymbol{Q}}$)*

For a commutative net $N = (T, P, C, w^-, w^+)$ with colour algebra $A_{\boldsymbol{Z}}$ consider the splitting

$$A_{\boldsymbol{Q}} \cong \Pi_{m\in \text{Specm } A\boldsymbol{Q}}\, A_m \text{ and } C_{\boldsymbol{Q}} \cong \Pi_{m\in \text{Specm } A\boldsymbol{Q}}\, C_m$$

from Theorem 2.7 and the induced splitting of markings

$$M = (M_m)_{m\in \text{Specm } A\boldsymbol{Q}} \in C^P(N, C_{\boldsymbol{Z}}).$$

i) We have the equivalence:

- A marking $M_{post} \in C^P(N, C_{\boldsymbol{Z}})$ is potentially reachable in the Petri net (N, M_{pre}) over $A_{\boldsymbol{Q}}$.
- Every marking component $M_{post,m}$, $m\in$ Specm $A_{\boldsymbol{Q}}$, satisfies the state equation over A_m

  $$M_{post,m} = M_{pre,m} + w_{T,Cm}(\tau_m)$$

  with respect to the incidence map at the level of C_m

  $$w_{T,Cm} := w_{T,Am} \otimes_{Am} id_{Cm}: C_T(N, C_m) \to C^P(N, C_m).$$

ii) If the colour algebra $A_{\boldsymbol{Z}}$ is reduced, then every incidence map at the level of C_m splits into a finite sum of morphisms between finite-dimensional vector spaces

$$w_{T,Am} : C_T(N, A_m) \to C^P(N, A_m).$$

In this case solvability of the state equation can be decided by the common rank criterion.

3 Potential reachability over $A_{\mathbf{Z}}$ in commutative nets

Extensions of $\mathbf{Z}$ like the ring of Gaussian integers $\mathbf{Z}$ [i] are the prototype of Dedekind domains ([AM1969], Chap. 9).

3.1 Definition (Dedekind domain)

A Noetherian domain A is called *Dedekind domain*, iff every non-zero prime ideal $p \in$ Spec A is maximal and every localization A_p, $p \in$ Spec A, is a principal ideal domain.

3.2 Theorem (Normalization of the colour algebra)

Consider a commutative net with reduced colour algebra $A_{\mathbf{Z}}$ and denote its normalization by

$$A_{\mathbf{Z}} \to B.$$

i) If $A_{\mathbf{Z}}$ has the irreducible components $(A_i)_{i=1,\dots,k}$, then B splits into a finite product of Dedekind domains

$$B = \Pi_{i=1,..,k} B_i,$$

and each B_i is the normalization of the corresponding component A_i.

ii) The extension $C_B := C_{\mathbf{Z}} \otimes_{A_{\mathbf{Z}}} B$ of the colour module $C_{\mathbf{Z}}$ of N to the normalization splits into a finite product

$$C_B = \Pi_{i=1,..,k} C_i,$$

and every C_i is a locally free module over B_i.

Proof. ad i) For the splitting of the normalization of a reduced affine algebra cf. [Bou1972], Chap. V, §1.2, Cor. 1 to Prop. 9. Because every component A_i is 1-dimensional, its normalization B_i is regular, hence a Dedekind domain.

ad ii) The splitting of the normalization B induces a corresponding splitting of the colour module C_B with factors $C_i = C_{\mathbf{Z}} \otimes_B B_i$, because the tensor product and the direct product commute. Every torsion-free finitely generated module E over a Dedekind domain D splits ([Bou1972], Chap. VII, § 4.10, Prop. 24) as

$$E = D^r \oplus I \text{ with } r \in \mathbf{N} \text{ and } I \subset D \text{ an ideal, QED.}$$

3.3 Theorem (Potential reachability over $A_{\mathbf{Z}}$)

For a commutative net $N = (T, P, C, w^-, w^+)$ with reduced colour algebra $A_{\mathbf{Z}}$ consider the normalization B and the splitting

$$B = \Pi_{i=1,..,k} B_i \text{ and } C_B = \Pi_{i=1,..,k} C_i$$

from Theorem 3.2. They induce for every marking $M \in C^P(N, C_Z)$ a splitting of the extension

$$M_B \in C^P(N, C_B) \text{ as } M_B = (M_i)_{i=1,..,k}.$$

i) If two markings M_{pre} and M_{post} satisfy the state equation over $A_{\mathbf{Z}}$, then all components $M_{pre,i}$ and $M_{post,i}$, i = 1,...,k, satisfy the state equation over the Dedekind domain B_i

$$M_{post,i} = M_{pre,i} + w_{T,C_i}(\tau_i)$$

with respect to the incidence map at the level of the colour module C_i

$$w_{T,Ci} = w_{T,C} \otimes_{AZ} B_i: C_T(N, C_i) \rightarrow C^P(N, C_i).$$

ii) Consider a Dedekind factor $D := B_i$ and set $C_D := C_i$. Then the incidenc map at the level of the colour module C_D

$$w_{T,D}: C_T(N, C_D) \rightarrow C^P(N, C_D)$$

splits into a finite direct sum of D-linear maps

$$w_D: C_T(N, D) \rightarrow C^P(N, D) \text{ resp. } w_I: C_T(N, I) \rightarrow C^P(N, I), I \subset D \text{ an ideal.}$$

Proof. ad ii) According to Theorem 3.2 the module C_i splits into the direct sum of a free D-module and an ideal I. This splitting carries over to the incidence map $w_{T,Ci}$, because it is induced from the incidence map $w_{T,Bi}$ at the level of the colour algebra, QED.

3.4 Theorem (Linear algebra over a Dedekind domain)

Consider a Dedekind domain A, an ideal $I \subset A$ and an A-linear map

$$f: I^n \rightarrow I^m$$

between locally free A-modules of finite rank. For a given element $y \in I^m$ we have the equivalence:

- There exists a solution $x \in I^n$ with $f (x) = y$
- rank (f) = rank (f, y) =: r and < minor [r, f] > = < minor [r, (f , y)] > ⊂ A.

Proof. Every A-linear endomorphism of I is a homothety, i.e.

$$Hom_A(I, I) \cong A.$$

Hence f is represented by a matrix

$$M(f) \in M(m \text{ x } n, A).$$

Every localization of a Dedekind domain is a principal ideal domain. Herefore the claim has been proven in [SW1999], Prop. 6.2. The Fitting condition implies, that the two A-modules

$$M_1 := f (I^n) \subset M_2 := M_1 + span_A< y >$$

have the same localization

$$M_{1,p} = M_{2,p}$$

for every prime ideal $p \in$ Spec A. The local-to-global principle ([AM1969], Prop. 3.9) implies $M_1 = M_2$, QED.

4 Conclusion and outlook

Commutative nets make a first non-trivial step venturing from p/t nets into the domain of coloured nets. Commutative nets are determined by their colour algebra, which has a rich additional structure in comparison with the ring ***Z***. The standard example of n dining philosophers has the colour algebra

$$A_{\mathbf{Z}} = \mathbf{Z} [t] / < t^n-1 >.$$

For n = 6 the polynomial t^6-1 splits into the four irreducible factors

$$\Phi_1(t) = t - 1, \Phi_2(t) = t + 1, \Phi_3(t) = t^2 + t + 1, \Phi_6(t) = t^2 - t + 1.$$

Figure 1 represents the colour algebra by the spectrum of its prime ideals. P/t nets have the colour algebra ***Z*** as represented by the horizontal axis. The study of p/t nets over the field ***Q*** focuses onto a single point, the origin of the axis. But already modulo-invariants and Proposition 1.2 take into consideration also the different primes of ***Z***: P/t nets have a 1-dimensional fine-structure over Spec ***Z***.

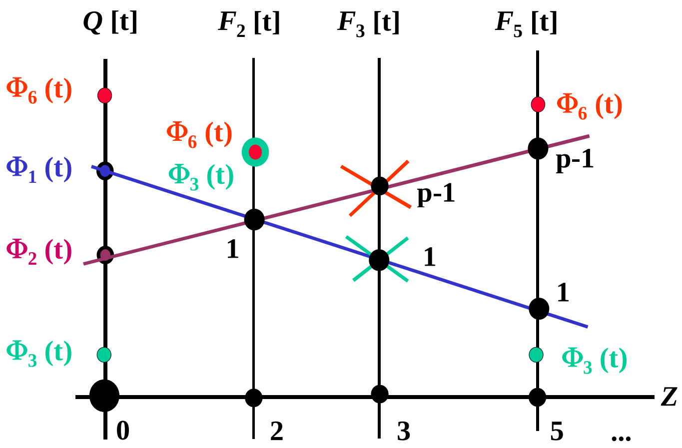

Figure 1 The colour algebra A_Z of 6 dining philosophers

Commutative nets open up a second dimension. Restriction to the vertical axis Spec ***Q***[t] considers the Artin structure of the net, cf. Chapter 2. Here the colour algebra factors into a product. In the reduced case one is left with finitely many p/t nets over number fields. But every point from Spec A_Q is the generic point of a whole irreducible component of Spec A_Z. In general the different components intersect in finitely many fibres Spec ***F***[t]. In the reduced case a first method to disentangle the components of Spec A_Z is the normalization, cf. Chapter 3.

In order to strengthen the necessary condition from Theorem 3.3 to a sufficient criterion, one has to focus on the finitely many singular points of A_Z. For every singular point one obtains an additional set of linear equations. The Parikh vectors on the components of the normalization have to satisfy these equations, in order to match to a global solution arising from A_Z.

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