Modules in Transition
Conservativity, Composition, and Colimits

Oliver Kutz[1] and Till Mossakowski[2]

[1] SFB/TR 8 Spatial Cognition, Bremen, Germany
okutz@informatik.uni-bremen.de

[2] DFKI Lab Bremen and SFB/TR 8 Spatial Cognition, Bremen, Germany
till.mossakowski@dfki.de

Abstract. Several modularity concepts for ontologies have been studied in the literature. Can they be brought to a common basis? We propose to use the language of category theory, in particular diagrams and their colimits, for answering this question. We outline a general approach for representing combinations of logical theories, or ontologies, through interfaces of various kinds, based on diagrams and the theory of institutions. In particular, we consider theory interpretations, language extensions, symbol identification, and conservative extensions. We study the problem of inheriting conservativity between sub-theories in a diagram to its colimit ontology. Finally, we apply this to the problem of conservativity when composing DDLs or $\mathcal{E}$-connections.

1 Introduction

In this paper, we propose to use the category theoretic notions of diagram and colimit in order to provide a common semantic backbone for various notions of modularity in ontologies.

At least three commonly used notions of 'module' in ontologies can be distinguished, depending of the kind of relationship between the 'module' and its supertheory (or superontology): (1) a module can be considered a 'logically independent' part within its superontology—this leads to the definition of module as a part of a larger ontology which is a conservative extensions of it; (2) a module can be a part of a larger 'integrated ontology', where the kind of integration determines the relation between the modules—this is the approach followed by modular ontology languages (e.g. DDLs, $\mathcal{E}$-connections etc.); (3) a 'part' of a larger theory can be considered a module for reasons of elegance, re-use, tradition, etc.—in this case, the relation between a module and its supertheory might be a language extension, theory extension/interpretation, etc.

The main contributions of the present paper are the following: (i) building on the theory of institutions, diagrams, and colimits, we show how these different notions of module can be considered simultaneously using the notion of a *module diagram*; (ii) we show how conservativity properties can be traced and inherited to the colimit of a diagram; (iii) we show how this applies to the composition problem in modular ontology languages such as DDLs and $\mathcal{E}$-connections.

In Section 2, we will introduce institutions and give several examples. Section 3 introduces the diagrammatic view of modules, and Section 4 studies the problem of conservativity in diagrams. Finally, in Section 5, we sketch heterogeneous diagrams, and apply this to modular ontology languages in Section 6.

2 Institutions

The study of modularity principles can be carried out to a quite large extent independently of the details of the underlying logical system that is used. The notion of **institutions** was introduced by Goguen and Burstall in the late 1970s exactly for this purpose (see Goguen and Burstall (1992)). They capture in a very abstract and flexible way the notion of a logical system by leaving open the details of signatures, models, sentences (axioms) and satisfaction (of sentences in models).

The importance of the notion of institutions lies in the fact that a surprisingly large body of logical notions and results can be developed in a way that is completely independent of the specific nature of the underlying institution.[1]

We assume some acquaintance with the basic notions of category theory, and refer to Adámek et al. (1990) for an introduction. The reader with no background in category theory can envisage a category as a "graph with composition of arrows", a functor as a "graph homomorphism". If $\mathcal{C}$ is a category, $\mathcal{C}^{op}$ is the dual category, where all arrows are reversed.

Definition 1. *An **institution** $I = (\mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \models)$ consists of*

- *a category* $\mathbf{Sign}$ *of* signatures*,*
- *a functor* $\mathbf{Sen}\colon \mathbf{Sign} \longrightarrow \mathbf{Set}$[2] *giving, for each signature* Σ*, the set of* sentences $\mathbf{Sen}(\Sigma)$*, and for each signature morphism* $\sigma\colon \Sigma \longrightarrow \Sigma'$*, the* sentence translation map $\mathbf{Sen}(\sigma)\colon \mathbf{Sen}(\Sigma) \longrightarrow \mathbf{Sen}(\Sigma')$*, where often* $\mathbf{Sen}(\sigma)(\varphi)$ *is written as* $\sigma(\varphi)$*,*
- *a functor* $\mathbf{Mod}\colon \mathbf{Sign}^{op} \longrightarrow \mathcal{CAT}$[3] *giving, for each signature* Σ*, the category of* models $\mathbf{Mod}(\Sigma)$*, and for each signature morphism* $\sigma\colon \Sigma \longrightarrow \Sigma'$*, the* reduct functor $\mathbf{Mod}(\sigma)\colon \mathbf{Mod}(\Sigma') \longrightarrow \mathbf{Mod}(\Sigma)$*, where often* $\mathbf{Mod}(\sigma)(M')$ *is written as* $M'|_\sigma$*,*
- *a satisfaction relation* $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$ *for each* $\Sigma \in |\mathbf{Sign}|$*,*

such that for each $\sigma\colon \Sigma \longrightarrow \Sigma'$ *in* $\mathbf{Sign}$ *the following* satisfaction condition *holds:*

$$(\star) \qquad M' \models_{\Sigma'} \sigma(\varphi) \text{ iff } M'|_\sigma \models_\Sigma \varphi$$

for each $M' \in |\mathbf{Mod}(\Sigma')|$ *and* $\varphi \in \mathbf{Sen}(\Sigma)$*, expressing that truth is invariant under change of notation and enlargement of context.* □

The only condition governing the behaviour of institutions is thus the *satisfaction condition* $(\star)$.[4]

A **theory** in an institution is a pair $T = (\Sigma, \Gamma)$ consisting of a signature $\mathsf{Sig}(T) = \Sigma$ and a set of Σ-sentences $\mathsf{Ax}(T) = \Gamma$, the axioms of the theory. The models of a theory T are those $\mathsf{Sig}(T)$-models that satisfy all axioms in $\mathsf{Ax}(T)$. Logical consequence is defined as usual: $T \models \varphi$ if all T-models satisfy φ. **Theory morphisms**, also called **interpretations of theories**, are signature morphisms

[1] For an extensive treatment of the model theory in this setting, see Diaconescu (2007).
[2] **Set** is the category having all sets as objects and functions as arrows.
[3] $\mathcal{CAT}$ is the category of categories and functors. Strictly speaking, $\mathcal{CAT}$ is not a category but only a so-called quasicategory, which is a category that lives in a higher set-theoretic universe.
[4] Note, however, that non-monotonic formalisms can only indirectly be covered this way, but compare, e.g., Guerra (2001).

that map axioms to logical consequences. An institution is **compact** if $T \models \varphi$ implies $T' \models \varphi$ for a finite subtheory T' of T.

Examples of institutions include:

Example 2. **First-order Logic.** In the institution $\mathbf{FOL}^{\mathsf{ms}=}$ of many-sorted first-order logic with equality, signatures are many-sorted first-order signatures, consisting of sorts and typed function and predicate symbols. Signature morphisms map symbols such that typing is preserved. Models are many-sorted first-order structures. Sentences are first-order formulas. Sentence translation means replacement of the translated symbols. Model reduct means reassembling the model's components according to the signature morphism. Satisfaction is the usual satisfaction of a first-order sentence in a first-order structure. □

Example 3. **Description Logics.** Signatures of the description logic $\mathcal{ALC}$ consist of a set of B of atomic concepts and a set R of roles, while signature morphisms provide respective mappings. Models are single-sorted first-order structures that interpret concepts as unary and roles as binary predicates. Sentences are subsumption relations $C_1 \sqsubseteq C_2$ between concepts, where concepts follow the grammar

$$C ::= B \mid \top \mid \bot \mid C_1 \sqcup C_2 \mid C_1 \sqcap C_2 \mid \neg C \mid \forall R.C \mid \exists R.C$$

Sentence translation and reduct is defined similarly as in $\mathbf{FOL}^{=}$. Satisfaction is the standard satisfaction of description logics. $\mathcal{ALC}^{\mathsf{ms}}$ is the many-sorted variant of $\mathcal{ALC}$. $\mathcal{ALCO}$ is obtained from $\mathcal{ALC}$ by extending signatures with nominals. The (sub-Boolean) description logic $\mathcal{EL}$ restricts $\mathcal{ALC}$ as follows: $C ::= B \mid \top \mid C_1 \sqcap C_2 \mid \exists R.C$. $\mathcal{SHOIN}$ extends $\mathcal{ALC}$ with role hierarchies, transitive and inverse roles, (unqualified) number restrictions, and nominals, etc. □

Example 4. **(Quantified) Modal Logics.** The modal logic $\mathbf{S4_u}$ has signatures as classical propositional logic, consisting of propositional variables. Sentences are built as in propositional logic, but add two unary modal operators, $\Box$ and $\blacksquare$. Models are standard Kripke structures but based on reflexive and transitive relations. Satisfaction is standard modal satisfaction, where $\Box$ is interpreted by the transitive reflexive relation, and $\blacksquare$ by universal quantification over worlds.

The standard formulation of first-order modal logic **QS5** (due to Kripke) has signatures similar to $\mathbf{FOL}^{=}$, including variables and predicate symbols. Sentences follow the grammar for $\mathbf{FOL}^{=}$-sentences using Booleans, quantifiers, and identity, while adding the $\Box$ operator, but leaving out constants and function symbols. Models are constant-domain first-order Kripke structures, with the usual first-order modal satisfaction. □

3 Modules as Diagrams

Several approaches to modularity in ontologies have been discussed in recent years, including the introduction of various so-called 'modular ontology languages'. The module system of the Web Ontology Language OWL itself is as simple as inadequate (Cuenca-Grau et al., 2006b): it allows for importing other ontologies, including cyclic imports. The language Casl (see Bidoit and Mosses, 2004; CoFI (The Common Framework Initiative), 2004) has been used in Lüttich et al. (2006) for ontologies. Beyond imports, it allows for renaming, hiding and parameterisation. Other languages that have been introduced include DDLs (Borgida and Serafini, 2002, 2003), $\mathcal{E}$-connections (Kutz et al., 2004), and P-DLs (Bao et al., 2006b,a), where, roughly speaking, more involved integration and modularisation mechanisms than plain imports are envisaged.

We will use the formalism of colimits of diagrams as a common semantic backbone for these languages.[5] The intuition behind this formalism is explained as follows:

> "Given a species of structure, say widgets, then the result of interconnecting a system of widgets to form a super-widget corresponds to taking the *colimit* of the diagram of widgets in which the morphisms show how they are interconnected." Goguen (1991)

The notion of **diagram** is formalised in category theory. Diagrams map an index category (via a functor) to a given category of interest. They can be thought of as graphs in the category. A **cocone** over a diagram is a kind of "tent": it consists of a tip, together with a morphism from each object involved in the diagram into the tip, such that the triangles arising from the morphisms in the diagram commute. A **colimit** is a universal, or minimal cocone. We refer to (Adámek et al., 1990) for formal details.

In the sequel, we will assume that the signature category has all finite colimits, which is a rather mild assumption; in particular, it is true for all the examples above. Moreover, we will rely on the fact that colimits of theories exist in this case as well; the colimit theory is defined as the union of all component theories in the diagram, translated along the signature morphisms of the colimiting cocone.

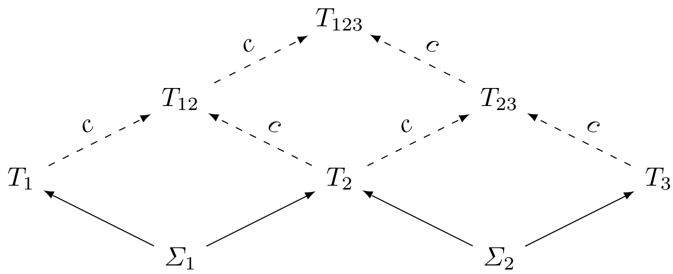
Definition 5. *A* ***module diagram*** *of ontologies is a diagram of theories such that the nodes are subdivided into* ontology nodes *and* interface nodes*.*

Composition of module diagrams is simply their union.

Example 6. Consider the union of the diagrams

where the Σ_i are interfaces and the T_i are ontologies. Think of e.g. T_{12} as being an ontology that imports T_1 and T_2, where Σ_1 contains all the symbols shared between T_1 and T_2. Then T_{12} (and T_{23}) can be obtained as pushouts, and so can the overall union T_{123} (which typically is then further extended with new concepts etc.). A "c" means "conservative"; this will be explained in Sect. 4.

[5] However, note that hiding is not covered by this approach.



It is clear that theories with an import structure are just tree-shaped diagrams, while both shared parts and cyclic imports lead to arbitrary graph-shaped diagrams. The translation of CASL (without hiding) to so-called development graphs detailed in (CoFI (The Common Framework Initiative), 2004) naturally leads to diagrams as well. Finally, the diagrams corresponding to modular languages like DDLs and $\mathcal{E}$-connections will be studied in Sect. 6. Thus, diagrams can be seen as a uniform mathematical formalism where properties of all of these module concepts can be studied. An important such property is conservativity.

4 Conservative Diagrams

Conservative diagrams are important because they imply that the combined ontology does not add new facts to the individual ontologies. Indeed, the notion of an ontology module of an ontology T has been defined as any "subontology T' such that T is a conservative extension of T'" (Ghilardi et al., 2006)—this perfectly matches our notion of conservative diagram below.

Definition 7. *A theory morphism $\sigma\colon T_1 \longrightarrow T_2$ is **proof-theoretically conservative**, if T_2 does not entail anything new w.r.t. T_1, formally, $T_2 \models \sigma(\varphi)$ implies $T_1 \models \varphi$. Moreover, $\sigma\colon T_1 \longrightarrow T_2$ is **model-theoretically conservative**, if any T_1-model M_1 has a σ-expansion to T_2, i.e. a T_2-model M_2 with $M_2|_\sigma = M_1$.*

It is easy to show that conservative theory morphisms compose. Moreover,

Proposition 8. *Model-theoretic implies proof-theoretic conservativity.*

However, the converse is not true in general:

Example 9. (Lutz and Wolter, 2007) Consider the following two $\mathcal{EL}$ TBoxes:

$$\Gamma_1 = \{\mathsf{Human} \sqsubseteq \exists \mathit{eats}.\top, \mathsf{Plant} \sqsubseteq \exists \mathit{grows\_in}.\mathsf{Area}, \mathsf{Vegetarian} \sqsubseteq \mathsf{Healthy}\}$$
$$\Gamma_2 = \{\mathsf{Human} \sqsubseteq \exists \mathit{eats}.\mathsf{Food}, \mathsf{Food} \sqcap \mathsf{Plant} \sqsubseteq \mathsf{Vegetarian}\}$$

It is easily seen that $\Gamma_1 \cup \Gamma_2$ is a proof-theoretic conservative extension of Γ_1 w.r.t. $\mathcal{EL}$. However, (Lutz and Wolter, 2007) also show this is not the case w.r.t. $\mathcal{ALC}$, as witnessed by

$$A := \mathsf{Human} \sqcap \forall \mathit{eats}.\mathsf{Plant} \sqsubseteq \exists \mathit{eats}.\mathsf{Vegetarian},$$

since $\Gamma_1 \cup \Gamma_2 \models A$, but $\Gamma_1 \not\models A$. In particular, it follows that $\Gamma_1 \cup \Gamma_2$ is not a model-theoretic conservative extension of Γ_1.

Definition 10. *A* ***(proof-theoretic, model-theoretic) conservative module diagram*** *of ontologies is a diagram of theories such that the theory morphism of any ontology node into the colimit of the diagram is (proof-theoretically resp. model-theoretically) conservative.*

By conservativity, the definition immediately yields:

Proposition 11. *The colimit ontology of a proof-theoretic (model-theoretic) module diagram is consistent (satisfiable)*[6] *if any of the component ontologies is.*

Thus, in particular, in a conservative module diagram, an ontology node O_i can only be consistent (satisfiable) if all other ontology nodes O_j, $j \neq i$, are consistent (satisfiable) as well.

The main question is how to ensure these conservativity properties in the united diagram. To study this, we introduce some notions from model theory, namely various notions of **interpolation** (for proof-theoretic conservativity) and **amalgamation** (for model-theoretic conservativity).

Craig interpolation plays a crucial role in connection with proof systems in structured theories, see Borzyszkowski (2002). It comes in various forms.

> **AIP.** (Arrow Interpolation) If $\models \varphi \to \psi$, then there exists some χ that only uses symbols occurring in both φ and ψ, with $\models \varphi \to \chi$ and $\models \chi \to \psi$.

However, this relies on a connective $\to$ being present in the institution. For the general study of module systems, the following formulation is more suitable:

> **TIP.** (Turnstile Interpolation) If $\varphi \models \psi$, then there exists some χ that only uses symbols occurring in both φ and ψ, with $\varphi \models \chi$ and $\chi \models \psi$.

TIP follows from AIP in the presence of a deduction theorem, and in this case, a further generalisation follows (see Areces and Marx, 1998):

> **SIP.** (Splitting Interpolation) If $\varphi_0, \varphi_1 \models \psi$, then there exists some χ that only uses symbols occurring in both ψ and the union of the symbols of φ_0 and φ_1, with $\varphi_0 \models \chi$ and $\varphi_1, \chi \models \psi$.

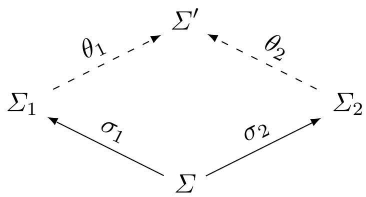
We now further generalise SIP in two ways. The first generalisation concerns the rather implicit use of signatures in the definitions above. Making this explicit would mean to assume that φ lives in a signature Σ_1, ψ lives in a signature Σ_2, the entailment $\varphi \models \psi$ lives in $\Sigma_1 \cup \Sigma_2$, and the interpolant in $\Sigma_1 \cap \Sigma_2$. Since we do not want to go into the technicalities for equipping an institution with unions and intersections (see Diaconescu et al. (1993) for details), we replace $\Sigma_1 \cap \Sigma_2$ with a signature Σ, and $\Sigma_1 \cup \Sigma_2$ with Σ' such that Σ' is obtained as a pushout from the other signatures via suitable signature morphisms (cf. the diagram below).

Secondly, we move from single sentences to *sets* of sentences. This is useful since we want to support DLs and TBox reasoning, and DLs like $\mathcal{EL}$ do not allow to rewrite 'conjunctions of subsumptions', i.e., we cannot collapse a TBox into a single sentence. (In case of compact logics, the use of sets is equivalent to the use of finite sets.)

We arrive at the following definition (in the sequel, fix an arbitrary institution $I = (\mathbf{Sign}, \mathbf{Sen}, \mathbf{Mod}, \models)$):

[6] Contrary to the terminology used in DL, we distinguish here proof-theoretic (syntactic) consistency of a theory T (which means $T \not\models \varphi$ for some sentence φ) from model-theoretic (semantic) satisfiability (which means $M \models T$ for some model M).

Definition 12. *The institution I has the* ***Craig-Robinson interpolation property*** *(Shoenfield (1967), Dimitrakos and Maibaum (2000)), if for any pushout*



any set Γ_1 *of* Σ_1*-sentences and any sets* Γ_2*,* Δ_2 *of* Σ_2*-sentences with*

$$\theta_1(\Gamma_1) \cup \theta_2(\Delta_2) \models \theta_2(\Gamma_2),$$

there exists a set of Σ*-sentences* Γ *(called the* interpolant*) such that*

$$\Gamma_1 \models \sigma_1(\Gamma) \text{ and } \Delta_2 \cup \sigma_2(\Gamma) \models \Gamma_2.$$

Craig-Robinson interpolation is, in general, strictly stronger than Craig interpolation. However, for almost all logics typically used in knowledge representation, they are indeed equivalent. We first give a criterion that applies to institutions generally, taken from Diaconescu (2007):

Proposition 13. *A compact institution with implication has Craig-Robinson interpolation iff it has Craig interpolation.*

Here, an institution has implication if for any two Σ-sentences φ, ψ, there exists a Σ-sentence χ such that, for any Σ-model M,

$$M \models \chi \text{ iff } (M \models \varphi \text{ implies } M \models \psi)$$

Since for modal logics, the deduction theorem (for the global consequence relation $\models$) generally fails, these logics do not have implication in the above sense, and we cannot apply Prop. 13. However, we can apply a slightly more concrete criterion for modal logics from the literature (cf. Prop. 2.1 in Areces and Marx (1998)):

Proposition 14. *Let* **L** *be a modal logic whose local consequence relation is compact and such that its class of Kripke frames is closed under point-generated subframes. Then Craig interpolation for* **L** *implies Craig-Robinson interpolation.*

Example 15 (Interpolation). The description logic $\mathcal{ALC}$ can be conceived as a syntactic variant of multi-modal **K**, for which (Gabbay, 1972; Gabbay and Maksimova, 2006) show Craig interpolation. **K** does not have implication, but satisfies the assumptions of Prop. 14. Hence, $\mathcal{ALC}$ has Craig-Robinson interpolation. The situation for DLs with nominals is less positive, in fact, the presence of nominals generally destroys (standardly formulated) Craig interpolation (compare the discussion in (Kutz, 2004, Chapter 3.4), and Areces and ten Cate (2006)) but can sometimes be restored, for instance, by treating nominals as logical constants, i.e., by freely reusing them. Here is a counterexample formulated in $\mathcal{ALCO}$. Let

$$\Gamma := \{\top \sqsubseteq \exists S.C \sqcap \exists S.\neg C\} \text{ and}$$
$$\Delta := \{\forall S.(D \sqcup i) \sqsubseteq \exists S.D\}$$

where i is a nominal. Clearly, $\Gamma \models \Delta$, for in every model $M \models \Gamma$, every point has at least two S-successors. But i can only be true in at most one of those successors, which entails $M \models \Delta$. Now, (using bisimulations) it can be shown that in $\mathcal{ALCO}$ there is no Δ' built from shared concept names alone (there are none) such that $\Gamma \models \Delta'$ and $\Delta' \models \Delta$. If we allow to use non-shared concept constructors (modalities), an interpolant could obviously be given in logics such as $\mathcal{SHOIN}$ by using (unqualified) number restrictions and by setting $\Delta' = \{\top \sqsubseteq (\geqslant 2S)\}$. Note, however, that ten Cate et al. (2006) show that interpolation still fails for $\mathcal{ALCQO}$ (since Beth fails), but that the Beth definability property is recovered for $\mathcal{ALCO}$@, or indeed for $\mathcal{SHIFO}$@.

Craig-Robinson for $\mathbf{FOL}^{\mathsf{ms}}$ is shown in Diaconescu (2007) (when one of the signature morphisms is injective on sorts). Craig interpolation for **FOL** is a classic result of Craig (1957), and Craig-Robinson follows since **FOL** is compact and has implication.

The failure of Craig interpolation for **QS5** is shown in Fine (1979).[7] But it holds for the quantified extension of **K** (Gabbay, 1972), and so does Craig-Robinson.

Finally, the modal logic $\mathbf{S4_u}$ has Craig-interpolation,[8] is compact (Goranko and Passy, 1992), and has implications (for $M \models \varphi \implies M \models \psi$, set $\chi = \blacksquare\varphi \rightarrow \blacksquare\psi$). Thus, $\mathbf{S4_u}$ has Craig-Robinson interpolation.

The amalgamation property (called 'exactness' in Diaconescu et al. (1993)) is a major technical assumption in the study of specification semantics, see Sannella and Tarlecki (1988).

Definition 16. *An institution I is **(weakly) semi-exact** if, for any pushout with notation as depicted above, given any pair $(M_1, M_2) \in \mathbf{Mod}(\Sigma_1) \times \mathbf{Mod}(\Sigma_2)$ that is* compatible *(in the sense that M_1 and M_2 reduce to the same Σ-model) can be* amalgamated *to a unique (or weakly amalgamated to a not necessarily unique) Σ'-model M (i.e., there exists a (unique) $M \in \mathbf{Mod}(\Sigma')$ that reduces to M_1 and M_2, respectively), and similarly for model morphisms.*

*I is (weakly) **exact**, if additionally the initial (= empty, usually) signature has exactly (at least) one model (and one model morphism).*

Note that (weak) exactness implies (weak) semi-exactness, while both are independent of interpolation.

Weak semi-exactness for these institutions follows with standard methods, see Diaconescu (2007). The same holds for exactness for the many-sorted variants. Exactness, however, obviously fails for the single-sorted logics as well as for **QS5**, because in these logics, the implicit universe resp. the implicit set of worlds leads to the phenomenon that the empty signature has many different models. The following propositions are folklore in institutional model theory, see Diaconescu (2007).

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[7] Craig interpolation for **QS5** can be restored, however, by extending the language with propositional quantifiers (Fitting, 2002) or nominals and @-operator (Areces et al., 2003).

[8] $\mathbf{S4_u}$ can be thought of as the independent fusion of the modal logics **S4** and **S5**, which both have interpolation, plus the *containment axiom* $\Box\varphi \rightarrow \blacksquare\varphi$. The interpolation property transfers to the fusion by a result of Kracht and Wolter (1991). However, since $\mathbf{S4_u}$ is a Sahlqvist axiomatisable logic whose frame conditions are universal Horn, it also follows for $\mathbf{S4_u}$ by a result of (Marx and Venema, 1997).

institution	weakly semi-exact	exact	Craig-Robinson
$\mathcal{EL}$	+	-	?
$\mathcal{ALC}^{\mathsf{ms}}$	+	+	+
$\mathcal{ALC}$	+	-	+
$\mathcal{ALCO}$	+	-	-
$\mathcal{ALCQO}$	+	-	-
$\mathcal{SHOIN}$	+	-	?
$\mathbf{FOL}^{\mathsf{ms}}$	+	+	+
QS5	+	-	-

Fig. 1. (Weak semi-) exactness and Craig-Robinson

Theorem 17. *1. In an institution with Craig-Robinson interpolation, proof-theoretic conservativity is preserved along pushouts.*
2. In an institution that is weakly semi-exact, model-theoretic conservativity is preserved along pushouts.

Call a diagram **connected** if the graph underlying its index category is connected when the identity arrows are deleted. A diagram is **thin**, or a **preorder**, if its index category is thin (i.e., there is at most one arrow between two given objects). A preorder is **finitely bounded inf-complete** if any two elements with a common lower bound have an infimum.

Theorem 18. *1. If an institution has Craig-Robinson interpolation, composition of module diagrams via union preserves proof-theoretic conservativity if the resulting diagram is a connected finitely bounded inf-complete preorder.*
2. If an institution is weakly semi-exact, composition of module diagrams via union preserves model-theoretic conservativity, if the resulting diagram is a connected finitely bounded inf-complete preorder. In case of weak exactness of the institution, the connectedness assumption can be replaced with satisfiability of the involved ontologies.

Proof. We can obtain colimits for arbitrary connected finitely bounded inf-complete preorders by successively taking pushouts. In each successive step, the pushout for two maximal nodes with a common lower bound is taken along the infimum, thereby decreasing the set of maximal nodes by one. Connectedness ensures that in case of more than one maximal node, at least two of them have a common lower bound. If a diagram with one maximal (=maximum) node is reached, this node provides the colimit.

Since in each individual module diagram, each ontology conservatively lies in the colimit of that module diagram, by the above argument, this carries over to the colimit of the union of the module diagrams.

In case of a weakly exact institution, we always can add the initial signature and the unique signature morphisms out of it, and thus get a connected diagram. Hence we can drop the connectedness assumption in this case. Instead, we need conservativity of the newly added signature morphisms, which amount to satisfiability of the ontologies. □

See Example 6 for a union of conservative diagrams.

5 Heterogeneous Module Diagrams

As Schorlemmer and Kalfoglou (2007) argue convincingly, relating ontologies may happen across different institutions, since ontologies are written in many different formalisms, like relation schemata, description logics, first-order logic, and modal logics.

Heterogeneous specification is based on some graph of logics and logic translations, formalised as institutions and so-called institution comorphisms, see Goguen and Roşu (2002). The latter are again governed by the satisfaction condition, this time expressing that truth is invariant also under change of notation across different logical formalisms:

$$M' \models^{J}_{\Phi(\Sigma)} \alpha_{\Sigma}(\varphi) \Leftrightarrow \beta_{\Sigma}(M') \models^{I}_{\Sigma} \varphi.$$

Here, $\Phi(\Sigma)$ is the translation of signature Σ from institution I to institution J, $\alpha_{\Sigma}(\varphi)$ is the translation of the Σ-sentence φ to a $\Phi(\Sigma)$-sentence, and $\beta_{\Sigma}(M')$ is the translation (or perhaps: reduction) of the $\Phi(\Sigma)$-model M' to a Σ-model.

The so-called *Grothendieck institution*, see Diaconescu (2002); Mossakowski (2002), is a technical device for giving a semantics to heterogeneous theories involving several institution. The Grothendieck institution is basically a flattening, or disjoint union, of a logic graph. A signature in the Grothendieck institution consists of a pair (L, Σ) where L is a logic (institution) and Σ is a signature in the logic L. Similarly, a Grothendieck signature morphism $(\rho, \sigma) : (L_1, \Sigma_1) \to (L_2, \Sigma_2)$ consists of a logic translation $\rho = (\Phi, \alpha, \beta)\colon L_1 \longrightarrow L_2$ plus an L_2-signature morphism $\sigma\colon \Phi(\Sigma_1) \longrightarrow \Sigma_2$. Sentences, models and satisfaction in the Grothendieck institution are defined in a component wise manner.

Hence, the definitions and results of the previous sections also apply to the heterogeneous case. Special care is needed in obtaining Craig-Robinson interpolation or weak semi-exactness in the Grothendieck institution. Diaconescu (2007) and Mossakowski (2006) contain some relevant results. In our experience with the heterogeneous tool set (see Mossakowski et al. (2007)), for the Grothendieck institution, it was much easier to obtain weak semi-exactness than Craig-Robinson interpolation.

6 Modular Languages

In this section, we show how the integration of ontologies via various 'modular languages' can be re-formulated in module diagrams. We concentrate on DDLs and $\mathcal{E}$-connections, which we reformulate as many-sorted theories. Finally, we analyse the problem of conservativity when composing DDLs or $\mathcal{E}$-connections. In the following, we will assume basic acquaintance with the syntax and semantics of both, DDLs and $\mathcal{E}$-connections. Details have to remain sketchy for lack of space.

6.1 DDLs and $\mathcal{E}$-connections as heterogeneous module diagrams

From the discussion of Section 5, it should be clear that DDLs or $\mathcal{E}$-connections can essentially be considered as many-sorted heterogeneous theories: component ontologies can be formulated in different logics, but have to be built from many-sorted vocabulary, and link relations are interpreted as relations connecting the sorts of the component logics (compare Baader and Ghilardi (2007) who note that this is an instance of a more general co-comma construction). To be more precise, assume a DDL $\mathfrak{D} = (\mathcal{S}_1, \mathcal{S}_2)$ is given. Knowledge bases for $\mathfrak{D}$ can contain **bridge rule** of the form:

$$C_i \xrightarrow{\sqsubseteq} C_j \quad \text{(into rule)} \qquad C_i \xrightarrow{\sqsupseteq} C_j \quad \text{(onto rule)}$$

where C_i and C_j are concepts from $\mathcal{S}_i$ and $\mathcal{S}_j$ $(i \neq j)$, respectively.[9]

An interpretation $\mathfrak{I}$ for a DDL knowledge base is a pair $(\{\mathcal{I}_i\}_{i \leq n}, \mathcal{R})$, where each $\mathcal{I}_i$ is a model for the corresponding $\mathcal{S}_i$, and $\mathcal{R}$ is a function associating with every pair (i, j), $i \neq j$, a binary relation $r_{ij} \subseteq W_i \times W_j$ between the domains W_i and W_j of $\mathcal{I}_i$ and $\mathcal{I}_j$, respectively.

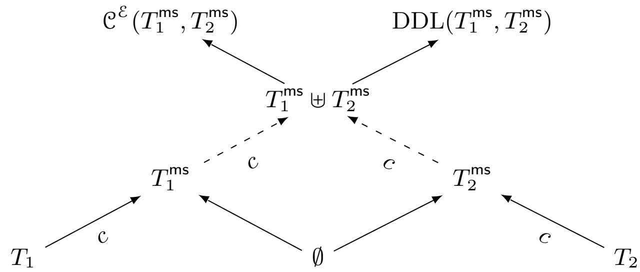


Fig. 2. $\mathcal{E}$-connections and DDLs many-sorted

In the many-sorted re-formulation of DDLs, the relation r_{ij} is now interpreted as a relation between the $\top$-sort of $\mathcal{S}_1$ and the $\top$-sort of $\mathcal{S}_2$. Bridge rules are expressed as existential restrictions of the form

$$(\sharp) \quad \exists r_{ij}.C_i \sqsubseteq C_j \qquad \text{and} \qquad \exists r_{ij}.C_i \sqsupseteq C_j$$

The fact that bridge rules are atomic statements in a DDL knowledge base now translates to a restriction on the grammar governing the usage of the link relation r_{ij} in the multi-sorted formalism (see Borgida (2007) for a discussion of related issues). In fact, the main difference between DDLs and various $\mathcal{E}$-connections now lies in the expressivity of this 'link language' $\mathcal{L}$ connecting the different sorts of the ontologies. In basic DDL as defined above, the only expressions allowed are those given in $(\sharp)$, so the link language of DDL is a certain, very weak sub-Boolean fragment of many sorted $\mathcal{ALC}$, namely the one given through $(\sharp)$. In $\mathcal{E}$-connections, expressions of the form $\exists r_{ij}.C_i$ are again concepts of $\mathcal{S}_j$, to which Booleans (or other operators) of $\mathcal{S}_j$ as well as restrictions using relations r_{ji} can be applied. Thus, the basic link language of $\mathcal{E}$-connections is sorted $\mathcal{ALCI}^{\mathsf{ms}}$(relative to the now richer languages of $\mathcal{S}_i$).[10]

[9] We consider here only DDL in its most basic form, comprising into and onto rules, but no individual correspondences. DDL in this form can be seen as a sub-formalism of one-way $\mathcal{E}$-connections, see (Cuenca-Grau and Kutz, 2007).

[10] But can be weakened to $\mathcal{ALC}^{\mathsf{ms}}$ or the link language of DDLs, or strengthened to more expressive many-sorted DLs such as $\mathcal{ALCQI}^{\mathsf{ms}}$.

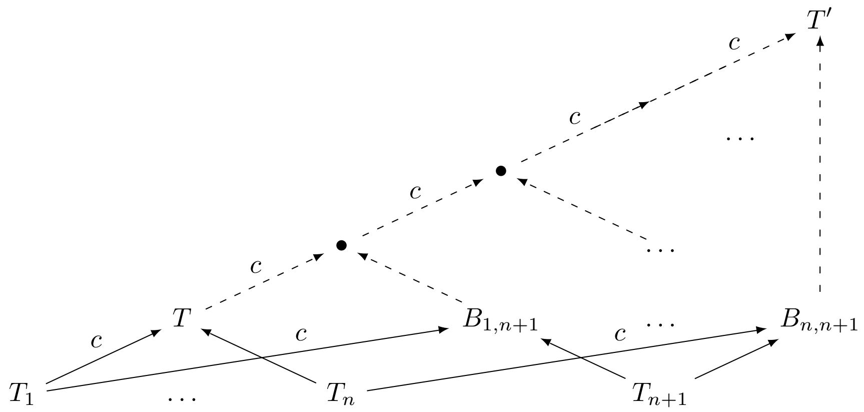
Such many-sorted theories can easily be represented in a diagram as shown in Figure 2. Here, we first (conservatively) obtain a disjoint union $T_1^{\mathsf{ms}} \uplus T_2^{\mathsf{ms}}$ as a pushout, where the component ontologies have been turned into sorted variants (using an institution comorphism from the single-sorted to the many-sorted logic), and the empty interface guarantees that no symbols are shared at this point. An $\mathcal{E}$-*connection* KB in language $\mathcal{C}^{\mathcal{E}}(T_1^{\mathsf{ms}}, T_2^{\mathsf{ms}})$ or a DDL KB in language $\mathrm{DDL}(T_1^{\mathsf{ms}}, T_2^{\mathsf{ms}})$ is then obtained as a (typically not conservative) theory extension.

6.2 The problem of iteration, or, how to keep modules intact

When connecting ontologies via bridges, or interfaces, this typically is not conservative everywhere, but only for some of the involved ontologies. We give a criterion for a single ontology to be conservative in the combination. While the theorem can be applied to arbitrary interface nodes, when applied to $\mathcal{E}$-connections or DDLs, we assume that bridge nodes contain DDL bridge rules or $\mathcal{E}$-connection assertions.

Theorem 19. *Assume that we work in an institution that has Craig-Robinson interpolation (is weakly semi-exact). Let ontologies $T_1, \ldots, T_n$ be connected via bridges B_{ij}, $i < j$. If T_i is proof-theoretically (model-theoretically) conservative in B_{ij} for $i < j$, then T_1 is proof-theoretically (model-theoretically) conservative in the resulting colimit ontology.*

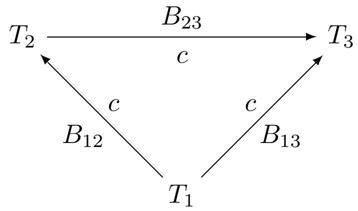
Proof. By induction over n. The base $n = 1$ is clear. Suppose now that the result holds for n, such that T_1 lies conservatively in the colimit ontology T, and we add T_{n+1} with corresponding bridges $B_{1,n+1}, \ldots, B_{n,n+1}$.



The resulting new colimit theory T' is constructed by successively constructing pushouts, whence we can use Theorem 17 to lift the conservativities of the morphisms $T_i \to T_{i,n+1}$ to conservativities of the arrows in the chain from T to T'. Since conservative theory morphisms compose, T_1 is conservative in T'. □

As concerns the applicability of the theorem, we have given an overview of logics being weakly semi-exact or having Craig-Robinson interpolation in Fig. 4. Of course, the conservativity assumptions have to be shown additionally. We here give an example of the failure of the claim of the theorem in case we work in a logic that lacks Craig-Robinson interpolation.

Example 20. Assume ontologies T_1, T_2, T_3 are formulated in the DL $\mathcal{ALCO}$, recall the example of failure of interpolation for $\mathcal{ALCO}$ given in Example 15, and consider the following diagram:



where $\mathsf{Sig}(T_1) \subseteq \{S, B, D, i\}$, $\mathsf{Sig}(T_2) \subseteq \{C_1, C_2\}$, and $\mathsf{Sig}(T_3) \subseteq \{B_1, B_2\}$. Moreover, $T_1 \supseteq \{\exists S.D\}$, and $B_{12} = \{\top \sqsubseteq \exists S.\exists R_1.C_1, \top \sqsubseteq \exists S.\exists R_1.\neg C_2\}$, $B_{13} = \{B_1 \equiv \exists R_3^{-1}.B, B_2 \equiv \exists R_3^{-1}.B\}$, and $B_{23} = \{C_1 \equiv \exists R_2.B_1, C_2 \equiv \exists R_2.B_2\}$.

Here, the roles R_1, R_2, R_3 can be seen as link relations, and since we apply existential restrictions $\exists S$ to $\exists R_2.C_1$ etc., the example can be understood as a composition of (binary) $\mathcal{E}$-connections.

The reader can check that T_i is conservative in B_{ij} for $j > i$. However, in the colimit (union) of this diagram, $\forall S.D \sqcup i \sqsubseteq \exists S.D$ follows, while this does not follow in T_1, and thus T_1 is not conservative in the colimit ontology.

7 Discussion and Outlook

Diagrams and their colimits offer the right level of abstraction to study conservativity issues in different languages for modular ontologies. We have singled out conditions that allow for lifting conservativity properties from individual diagrams to their combinations.

An interesting point is the question whether proof-theoretic or model-theoretic conservativity should be used. The model-theoretic notion ensures 'modularity' in more logics than the proof-theoretic one since the lifting theorem for the former only depends on mild amalgamation properties. By contrast, for the latter one needs Craig-Robinson interpolation which fails, e.g., for some description logics with nominals, and also for **QS5**—but these logics are used in practice for ontology design.

Moreover, as argued in Sect. 5, when relating ontologies across different institutions, the model-theoretic notion is more feasible. Finally, it has the advantage of being independent of the particular language, which implies avoidance of examples like the one presented in Lutz and Wolter (2007) (Example 9 above), where a given ontology extension is proof-theoretically conservative in $\mathcal{EL}$ but not in $\mathcal{ALC}$. Of course, model-theoretic conservativity generally is harder to decide, but it can be ensured by syntactic criteria, and the work related to this is promising (e.g. Cuenca-Grau et al., 2006a, 2007).

Acknowledgements

Work on this paper has been supported by the Vigoni program of the DAAD and by the DFG-funded collaborative research centre 'Spatial Cognition'.

We thank John Bateman, Joana Hois, and Lutz Schröder for fruitful discussions.

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