Concept Graphs as Semantic Structures for Contextual Judgment Logic

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Abstract. This paper presents a mathematization of the *philosophical doctrine of judgments* as an extension of the mathematization of the philosophical doctrine of concepts developed in Formal Concept Analysis. The chosen approach was strongly stimulated by J. F. Sowa's theory of *conceptual graphs*. The mathematized conceptual graphs, called *concept graphs*, are mathematical semantic structures based on formal contexts and their formal concepts; those semantic structures are viewed as formal judgments in the underlying *Contextual Judgment Logic*. In this paper concept graphs are systematically built up starting with *simple concept graphs* in section 2 and continuing with *existential concept graphs* in section 3, with *implicational and clausal concept graphs* in section 4, and finally with generalizations of concept graphs in section 5. Examples are illustrating the different types of concept graphs.

Contents

1 Semantic Structures for Contextual Judgment Logic

"Contextual Logic" has grown out of attemps to "restructure" lattice theory and mathematical logic (see [Wi82], [Wi96], [Wi97], [Pr98]). These attempts were stimulated by the german scholar Hartmut von Hentig with his charge to *restructure scientific disciplines* which he explains as follows:

> "The restructuring of scientific disciplines within themselves become more and more necessary to make them better learnable, mutually available, and criticizable in more general surroundings, also beyond disciplinary competence. This restructuring may and must be performed by general patterns of perceptions, thought, and action of our civilization." ([He74], p.33f.)
> For this: "Sciences have to examine their disciplinarity, and this means: to uncover their unconscious purposes, to declare their conscious purposes, to select and to adjust their means according to those purposes, to explain possible consequences comprehensibly and publicly, and to make accessible their ways of scientific finding and their results by the every-day language." ([He74], p.136f.)

Restructuring lattice theory has been started in 1979 by mathematizing concepts and concept hierarchies which led to the notions of "formal context" and "concept lattice" (see [Wi82]). A *formal context* was defined as a triple (G, M, I) where G is a set, the elements of which are called "objects", M is a set, the elements of which are called "attributes", and $I \subseteq G \times M$ is a binary relation for which $(g, m) \in I$ (also written: gIm) is read: "the object g has the attribute m". A *formal concept* of (G, M, I) was then defined as a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ satisfying:

$$A = \{g \in G \mid \forall m \in B : gIm\}(=: B') \text{ and } B = \{m \in M \mid \forall g \in A : gIm\}(=: A').$$

A and B are called the *extent* and the *intent* of the formal concept (A, B), respectively. The set $\mathfrak{B}(G, M, I)$ of all formal concepts of a given formal context (G, M, I) carries an order relation $\leq$ defined by $(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2$ $(\Leftrightarrow B_1 \supseteq B_2)$ so that $(\mathfrak{B}(G, M, I), \leq)$ becomes a complete lattice, the so-called *concept lattice* of (G, M, I) which is structured by the following $\bigwedge-$operation and $\bigvee-$operation:

$$\bigwedge_{t\in T}(A_t, B_t) := (\bigcap_{t\in T} A_t, (\bigcup_{t\in T} B_t)''), \qquad \bigvee_{t\in T}(A_t, B_t) := ((\bigcup_{t\in T} A_t)'', \bigcap_{t\in T} B_t).$$

A comprehensive introduction to the rich theory of concept lattices is presented in the monograph *"Formal Concept Analysis: Mathematical Foundation"* [GW99a].

Restructuring mathematical logic has been started in the early 1990s and first presented at the conference on *"logic and algebra"* held in Pontignano (Siena) in April 1994 (see [Wi96]). The restructuring approach was based on the traditional philosophical logic which is founded on "the three essential main functions of thinking - *concepts*, *judgments*, and *conclusions*" ([Ka88], p.6) and therefore, on the elementary level, presented in three parts: the doctrine of concepts, the doctrine of judgments, and the doctrine of conclusions. These doctrines are essential, since human thinking is based on concepts as basic units of thought, on judgments as assertional combinations of concepts, and on conclusions as entailments between judgments. For mathematizing concepts, judgments, and conclusions, they shall be primarily understood as *semantic structures* which are basic for logical thinking.

Mathematizing the doctrine of concepts, using a contextual approach, has been already performed to a great extent in developing *Formal Concept Analysis* (cf. [GW99a], in particular: [GW99b],[Wi00]),[Ga05]). Therefore, this paper continues to present the *mathematization of the doctrine of judgments* which builds up a *Contextual Judgment Logic* based on developments in Formal Concept Analysis (cf. [Wi01],[Wi03]). The chosen approach was strongly stimulated by J. F. Sowa's theory of conceptual graphs [So84] since those graphs can be understood as semantic structures which represent logical judgments. The mathematized conceptual graphs, called *concept graphs*, are mathematical semantic structures based on formal contexts and their formal concepts (cf. [Wi97],[Wi02]); those semantic structures are considered as formal judgments in the underlying Contextual Judgment Logic. How concept graphs can be systematically introduced and analysed is described in the next sections: *simple concept graphs* in section 2, *existential concept graphs* in section 3, *implicational and clausal concept graphs* in section 4, and generalizations of concept graphs, in particular *concept graphs with local negations* in section 5.

2 Simple Concept Graphs and Their Conceptual Contents

Each step of the presented development of concept graphs shall start with an example of a judgment represented graphically by a conceptual graph as standardized by John Sowa (cf. [So92]). Those judgments are deduced from the following statement written by Charles S. Peirce ([Pe92], p.114):

> "Mathematics ... is the only one of the sciences which does not concern itself to inquire what the actual facts are, but studies hypotheses exclusively."

To obtain an example of a *simple conceptual graph*, we consider the judgment: "The science mathematics studies the hypothesis $2^{\aleph_0} = \aleph_1$" (called "continuum hypothesis"). This judgment may be represented by the simple conceptual graph shown in Fig. 1. In that graph, "science" and "hypothesis" name concepts, while "mathematics" and "$2^{\aleph_0} = \aleph_1$" name objects which fall under the concepts "science" and "hypothesis", respectively; furthermore, the relational concept "study" links the science "mathematics" with the hypothesis "$2^{\aleph_0} = \aleph_1$".

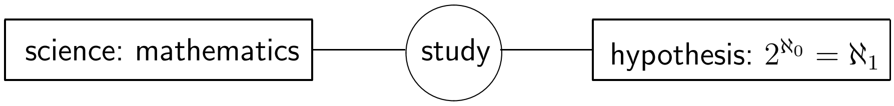

Fig. 1. Example of a simple conceptual graph

The example shows that judgments may join plain concepts with relational concepts so that a mathematization of judgments has to offer besides formal concepts also "relation concepts". How this has been performed and further developed shall be explained in the rest of this section (cf. [Wi04], pp. 53 – 55).

A *power context family* is a sequence $\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \mathbb{K}_2, \ldots)$ of formal contexts $\mathbb{K}_k := (G_k, M_k, I_k)$ with $G_k \subseteq (G_0)^k$ for $k = 1, 2, \ldots$. The formal concepts of $\mathbb{K}_k$ with $k = 1, 2, \ldots$ are called *relation concepts*, because they represent k-ary relations on the object set G_0 by their extents.

A *relational graph* is a structure (V, E, ν) consisting of two disjoint sets V and E together with a map $\nu : E \to \bigcup_{k=1,2,\ldots} V^k$; the elements of V and E are called *vertices* and *edges*, respectively, and $\nu(e) = (v_1, \ldots, v_k)$ is read: $v_1, \ldots, v_k$ are the *adjacent vertices* of the *k-ary edge* e ($|e| := k$ is the *arity* of e; the arity of a vertex is defined to be 0). Let $E^{(k)}$ be the set of all elements of $V \cup E$ of arity k ($k = 0, 1, 2, \ldots$).

A *simple concept graph* of a power context family $\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \mathbb{K}_2, \ldots)$ with $\mathbb{K}_k := (G_k, M_k, I_k)$ for $k = 0, 1, 2, \ldots$ is a structure $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ for which

- (V, E, ν) is a relational graph,
- $\kappa : V \cup E \to \bigcup_{k=0,1,2,\ldots} \underline{\mathfrak{B}}(\mathbb{K}_k)$ is a mapping such that $\kappa(u) \in \underline{\mathfrak{B}}(\mathbb{K}_k)$ for all $u \in E^{(k)}$,
- $\rho : V \to \mathfrak{P}(G_0)\backslash\{\emptyset\}$ is a mapping such that $\rho(v) \subseteq Ext(\kappa(v))$ for all $v \in V$ and, furthermore, $\rho(v_1) \times \cdots \times \rho(v_k) \subseteq Ext(\kappa(e))$ for all $e \in E$ with $\nu(e) = (v_1, \ldots, v_k)$;
- in general, $Ext(\mathfrak{c})$ denotes the extent of the formal concept $\mathfrak{c}$.

It is convenient to consider the mapping ρ not only on vertices but also on edges: for all $e \in E$ with $\nu(e) = (v_1, \ldots, v_k)$, let $\rho(e) := \rho(v_1) \times \cdots \times \rho(v_k)$.

A *subgraph* of a concept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ is a concept graph $\mathfrak{G}_s := (V_s, E_s, \nu_s, \kappa_s, \rho_s)$ for which $V_s \subseteq V$, $E_s \subseteq E$, $\nu_s = \nu|_{E_s}$, $\kappa_s = \kappa|_{V_s \cup E_s}$, and $\rho_s = \rho|_{V_s}$. The *union* and *intersection* of subgraphs $\mathfrak{G}_t := (V_t, E_t, \nu_t, \kappa_t, \rho_t)$ $(t \in T)$ of a concept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ are defined by

$$\bigcup_{t \in T} \mathfrak{G}_t := (\bigcup_{t \in T} V_t, \bigcup_{t \in T} E_t, \bigcup_{t \in T} \nu_t, \bigcup_{t \in T} \kappa_t, \bigcup_{t \in T} \rho_t),$$

$$\bigcap_{t \in T} \mathfrak{G}_t := (\bigcap_{t \in T} V_t, \bigcap_{t \in T} E_t, \bigcap_{t \in T} \nu_t, \bigcap_{t \in T} \kappa_t, \bigcap_{t \in T} \rho_t).$$

Lemma 1 *The union and intersection of subgraphs of a concept graph $\mathfrak{G}$ is always a subgraph of $\mathfrak{G}$ again.*

From the background knowledge coded in a power context family $\vec{\mathbb{K}}$, two types of material inferences shall be made formally explicit: Let $k = 0, 1, 2, \ldots$;

1. *object implications*: for $A, C \subseteq G_k$, $\mathbb{K}_k$ satisfies $A \to C$ if $A^{I_k} \subseteq C^{I_k}$ and,
2. *concept implications*: for $\mathfrak{B}, \mathfrak{D} \subseteq \mathfrak{B}(\mathbb{K}_k)$, $\mathbb{K}_k$ satisfies $\mathfrak{B} \to \mathfrak{D}$ if $\bigwedge \mathfrak{B} \leq \bigwedge \mathfrak{D}$.

The *formal implications* $A \to C$ and $\mathfrak{B} \to \mathfrak{D}$ give rise to a closure system $\mathcal{C}(\mathbb{K}_k)$ on $\mathbb{S}^{imp}(\mathbb{K}_k) := \{(g, \mathfrak{b}) \in G_k \times \mathfrak{B}(\mathbb{K}_k) \mid g \in Ext(\mathfrak{b})\}$ consisting of all subsets Y of $\mathbb{S}^{imp}(\mathbb{K}_k)$ which have the following property:

(P_k) If $A \times \mathfrak{B} \subseteq Y$ and if $\mathbb{K}_k$ satifies $A \to C$ and $\mathfrak{B} \to \mathfrak{D}$ then $C \times \mathfrak{D} \subseteq Y$.

For $k = 1, 2, \ldots$, the $\mathbb{K}_k$-*conceptual content* $C_k(\mathfrak{G})$ of a concept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ of a power context family $\vec{\mathbb{K}}$ is defined as the closure of

$$\{(\vec{g}, \kappa(e)) \mid e \in E^{(k)} \text{ and } \vec{g} \in \rho(e)\}$$

with respect to the closure system $\mathcal{C}(\mathbb{K}_k)$;
the $\mathbb{K}_0$-*conceptual content* $C_0(\mathfrak{G})$ of $\mathfrak{G}$ is defined as the closure of

$$\{(g, \kappa(v)) \mid v \in V \text{ and } g \in \rho(v)\} \cup$$
$$\{(g_i, (G_0, G_0^{I_0})) \mid \exists((g_1, \ldots, g_k), \mathfrak{c}) \in C_k(\mathfrak{G}) \text{ with } g_i \in \{g_1, \ldots, g_k\}\}$$

with respect to the closure system $\mathcal{C}(\mathbb{K}_0)$. Then

$$C(\mathfrak{G}) := C_0(\mathfrak{G}) \,\dot\cup\, C_1(\mathfrak{G}) \,\dot\cup\, C_2(\mathfrak{G}) \,\dot\cup\, \ldots$$

is called the ($\vec{\mathbb{K}}$-)*conceptual content* of the concept graph $\mathfrak{G}$.

The defined conceptual contents give rise to an *information* (*quasi-*) *order* $\lesssim$ on the set of all concept graphs of a power context family: A concept graph $\mathfrak{G}_1 := (V_1, E_1, \nu_1, \kappa_1, \rho_1)$ is said to be *less informative* (*more general*) than a concept graph $\mathfrak{G}_2 := (V_2, E_2, \nu_2, \kappa_2, \rho_2)$ (in symbols: $\mathfrak{G}_1 \lesssim \mathfrak{G}_2$) if

$$C_k(\mathfrak{G}_1) \subseteq C_k(\mathfrak{G}_2) \text{ for } k = 0, 1, 2, \ldots;$$

$\mathfrak{G}_1$ and $\mathfrak{G}_2$ are called *equivalent* (in symbols: $\mathfrak{G}_1 \sim \mathfrak{G}_2$) if $\mathfrak{G}_1 \lesssim \mathfrak{G}_2$ and $\mathfrak{G}_2 \lesssim \mathfrak{G}_1$ (i.e., $C_k(\mathfrak{G}_1) = C_k(\mathfrak{G}_2)$ for $k = 0, 1, 2, \ldots$). The set of all equivalence classes of concept graphs of a power context family $\vec{\mathbb{K}}$ together with the order induced by the quasi-order $\lesssim$ is a *complete lattice* denoted by $\widetilde{\Gamma}(\vec{\mathbb{K}})$.

3 Existential Concept Graphs and Their Conceptual Contents

To obtain an example of an *existential conceptual graph*, we modify the judgment of section 2 as follows: "The science mathematics studies hypotheses". Logically equivalent is the judgment: "There exists at least one hypothesis studied by the science of mathematics". This judgment may be represented by the existential conceptual graph shown in Fig. 2.

The example shows that judgments may embody existentially quantified variables which are usually indicated by letters like x, y, z (sometimes they are replaced by a so-called "coreference link"). The mathematization of existential conceptual graphs whose variables are from a variable set X can be based on "free X-extensions" of a power context family. Such mathematization generalizes the approach of section 2 so that it becomes a wider range of applications (cf. [Wi04], pp. 55 – 57).

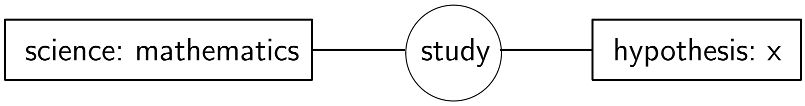

Fig. 2. Example of an existential conceptual graph

For a set X of variables, an X-*interpretation* into a set G_0 with $G_0 \cap X = \emptyset$ is defined as a mapping $\chi : G_0 \cup X \to G_0$ with $\chi(g) = g$ for all $g \in G_0$; the set of all X-interpretations into G_0 is denoted by $B(X, G_0)$. The *free X-extension* of the power context family $\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \mathbb{K}_2, \ldots)$ with $\mathbb{K}_k := (G_k, M_k, I_k)$ for $k = 0, 1, 2, \ldots$ and $G_0 \cap X = \emptyset$ is defined as a power context family $\vec{\mathbb{K}}[X] := (\mathbb{K}_0[X], \mathbb{K}_1[X], \mathbb{K}_2[X], \ldots)$ for which

- $\mathbb{K}_0[X] := (G_0[X], M_0[X], I_0[X])$ with $G_0[X] := G_0 \cup X$,
  $M_0[X] := M_0$, $I_0[X] := I_0 \cup (X \times \{m \in M_0 \mid \{m\}^{I_0} \neq \emptyset\})$,
- $\mathbb{K}_k[X] := (G_k[X], M_k[X], I_k[X])$ $(k = 1, 2, \ldots)$ with
  $G_k[X] := \{(u_1, \ldots, u_k) \in G_0[X]^k \mid \exists \chi \in B(X, G_0) : (\chi(u_1), \ldots, \chi(u_k)) \in G_k\}$,
  $M_k[X] := M_k$, and
  $(u_1, \ldots, u_k) I_k[X] m :\Longleftrightarrow \exists \chi \in B(X, G_0) : (\chi(u_1), \ldots, \chi(u_k)) I_k m$.

$\vec{\mathbb{K}}[X]$ is called an *existential power context family.*

For defining existential concept graphs, the surjective $\bigwedge$-homomorphisms $\pi_k^X : \underline{\mathfrak{B}}(\mathbb{K}_k[X]) \to \underline{\mathfrak{B}}(\mathbb{K}_k)$ $(k = 0, 1, 2, \ldots)$ are needed which are determined by

$$\pi_k^X(A, B) := (A \cap G_k, (A \cap G_k)^{I_k}) \text{ for } (A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_k[X]).$$

An *existential concept graph* of a power context family $\vec{\mathbb{K}}$ is defined as a concept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ of a free X-extension $\vec{\mathbb{K}}[X]$ for which an X-interpretation χ into G_0 exists such that $\mathfrak{G}^\chi := (V, E, \nu, \kappa^\chi, \rho^\chi)$ with $\kappa^\chi(u) := \pi_k^X(\kappa(u))$ and $\rho^\chi(v) := \chi(\rho(v))$ is a concept graph of $\vec{\mathbb{K}}$; χ is then called an X-interpretation *admissible on* $\mathfrak{G}$. For a fixed variable set X, $\mathfrak{G}$ is more precisely named an existential concept graph of $\vec{\mathbb{K}}$ *over* X.

Lemma 2 *The subgraphs of an existential concept graph over X are existential concept graphs over X, too.*

The *conceptual content* of an existential concept graph $\mathfrak{G}_X$ of a power context family $\vec{\mathbb{K}}$ is defined as the conceptual content of $\mathfrak{G}_X$ understood as a concept graph of the free X-extension $\vec{\mathbb{K}}[X]$. To clarify this, it is helpful to show how variables give rise to object implications of the relational contexts $\mathbb{K}_k[X]$ as indicated in the following lemma:

Lemma 3 *Let $\mathbb{K}_k[X] := (G_k[X], M_k[X], I_k[X])$ with $k \in \{1, 2, \ldots\}$ be a relational context of an existential power context family $\vec{\mathbb{K}}[X]$; furthermore, let α be a map of $G_0 \cup X$ into itself satisfying $\alpha(g) = g$ for all $g \in G_0$. Then $\mathbb{K}_k[X]$ has the object implications $\{(\alpha(u_1), \ldots, \alpha(u_k))\} \longrightarrow \{(u_1, \ldots, u_k)\}$ with $u_1, \ldots, u_k \in G_0 \cup X$.*

For a permutation π of the variable set X, let α_π be the map of $G_0 \cup X$ into itself with $\alpha_\pi(g) = g$ for all $g \in G_0$ and $\alpha_\pi(x) = \pi(x)$ for all $x \in X$. Then we obtain the object implication $\{(\alpha_\pi(u_1), \ldots, \alpha_\pi(u_k))\} \longrightarrow \{(u_1, \ldots, u_k)\}$ with $u_1, \ldots, u_k \in G_0 \cup X$. Together with the corresponding object implication for π^{-1}, this yields that changing variables according to a permutation of X in a (k-ary) object of $\mathbb{K}_k[X]$ does not change the intension of that object.

An existential concept graph $\mathfrak{G}_1 := (V_1, E_1, \nu_1, \kappa_1, \rho_1)$ is said to be *less informative* (*more general*) than $\mathfrak{G}_2 := (V_2, E_2, \nu_2, \kappa_2, \rho_2)$ (in symbols: $\mathfrak{G}_1 \lesssim \mathfrak{G}_2$) if

$C_k(\mathfrak{G}_1) \subseteq C_k(\mathfrak{G}_2)$ for $k = 0, 1, 2, \ldots$; $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are called *equivalent* (in symbols: $\mathfrak{G}_1 \sim \mathfrak{G}_2$) if $\mathfrak{G}_1 \lesssim \mathfrak{G}_2$ and $\mathfrak{G}_2 \lesssim \mathfrak{G}_1$ (i.e., $C_k(\mathfrak{G}_1) = C_k(\mathfrak{G}_2)$ for $k = 0, 1, 2, \ldots$). The set of all equivalence classes of existential concept graphs of a power context family $\vec{\mathbb{K}}$ over a fixed set X of variables together with the order induced by the quasi-order $\lesssim$ is an *ordered set* denoted by $\widetilde{\Gamma}(\vec{\mathbb{K}}; X)$.

4 Implicational and Causal Concept Graphs

For representing exactly Peirce's judgment "mathematics studies hypotheses exclusively", we have to generalize existential conceptual graphs further to *implicational conceptual graphs*. This becomes clear when we consider an equivalent formulation of Peirce's judgment, namely: "If mathematics studies a proposition then mathematics studies a hypothesis". A representation of this judgment by an implicational conceptual graph is pictured in Fig. 3.

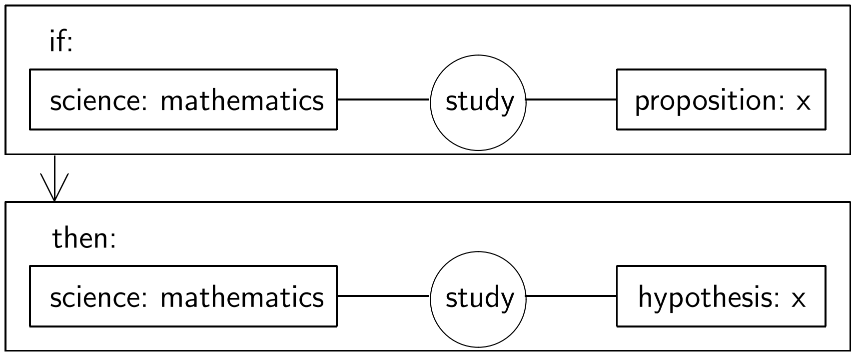

Fig. 3. Example of an implicational conceptual graph

The example shows an implicational judgment in which the premise and the conclusion contain the same variable x; this indicates that the proposition x is, more precisely, a hypothesis. The mathematization of implicational conceptual graphs who are composed by two subgraphs representing a premise and a corresponding conclusion, respectively, can be viewed as a generalization of existential concept graphs (cf. [Wi04], pp. 57 – 59).

An *implicational concept graph* of a power context family $\vec{\mathbb{K}}$ is defined as an existential concept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ of $\vec{\mathbb{K}}$ over a variable set X with a designated pair $(p\mathfrak{G}, c\mathfrak{G})$ of subgraphs such that

1. $\mathfrak{G}$ is the union of $p\mathfrak{G}$ and $c\mathfrak{G}$, and
2. each X-interpretation admissible on $p\mathfrak{G}$ is also admissible on $c\mathfrak{G}$ (and hence on $\mathfrak{G}$ too).

$p\mathfrak{G} \to c\mathfrak{G}$ may be written instead of $\mathfrak{G}$; the subgraphs $p\mathfrak{G}$ and $c\mathfrak{G}$ are called the *premise* and the *conclusion*, resp.

For an existential concept graph $\overline{\mathfrak{G}}$ of a power context family $\vec{\mathbb{K}}$ over a variable set X, the formal context $\mathbb{K}(X; \overline{\mathfrak{G}}) := (B(X, G_0), Sub(\overline{\mathfrak{G}}), \triangleright)$ is defined where

– the object set $B(X, G_0)$ consists of all X-interpretations into the object set G_0 of the formal context $\mathbb{K}_0$ in $\vec{\mathbb{K}}$,
– the attribute set $Sub(\overline{\mathfrak{G}})$ is the set of all subgraphs of $\overline{\mathfrak{G}}$,
– $\chi \triangleright \mathfrak{G}$ means that the X-interpretation χ is admissible on the subgraph $\mathfrak{G}$ of $\overline{\mathfrak{G}}$.

Proposition 1 *$\{\mathfrak{G}_s \mid s \in S\} \to \{\mathfrak{G}_t \mid t \in T\}$ is an attribute implication of $\mathbb{K}(X;\overline{\mathfrak{G}})$ if and only if $\bigcup_{s\in S} \mathfrak{G}_s \to \bigcup_{t\in T} \mathfrak{G}_t$ is an implicational concept graph of $\vec{\mathbb{K}}$ over X.*

Proposition 2 *$\mathbb{K}(X;\overline{\mathfrak{G}}) := (B(X, G_0), Sub(\overline{\mathfrak{G}}), \rhd)$ is always a formal context of which all extents are non-empty attribute extents. Conversely, let $\mathbb{K} := (G, M, I)$ be a clarified formal context of which all extents are non-empty attribute extents; then $\mathbb{K}$ is isomorphic to the clarified context of the formal context $\mathbb{K}(\{x\};\overline{\mathfrak{G}}) := (B(\{x\}, G), Sub(\overline{\mathfrak{G}}), \rhd)$ where $\overline{\mathfrak{G}} := (V, E, \nu, \kappa, \rho)$ is the existential concept graph of the power context family $\vec{\mathbb{K}} := (\mathbb{K})$ over $\{x\}$ with $V := M$, $E := \emptyset$, $\nu := \emptyset$, $\kappa(m) := \mu m$, and $\rho(m) := \{x\}$.*

Corollary 1 *The concept lattices $\underline{\mathfrak{B}}(\mathbb{K}(X;\overline{\mathfrak{G}}))$ are up to isomorphism the concept lattices of formal contexts.*

Implicational conceptual graphs can even be generalized to clausal conceptual graphs in which the conclusion consists of a disjunction of propositions (cf. [Wi04], pp. 59 – 60). An example of a clausal conceptual graph is shown in Fig. 4.

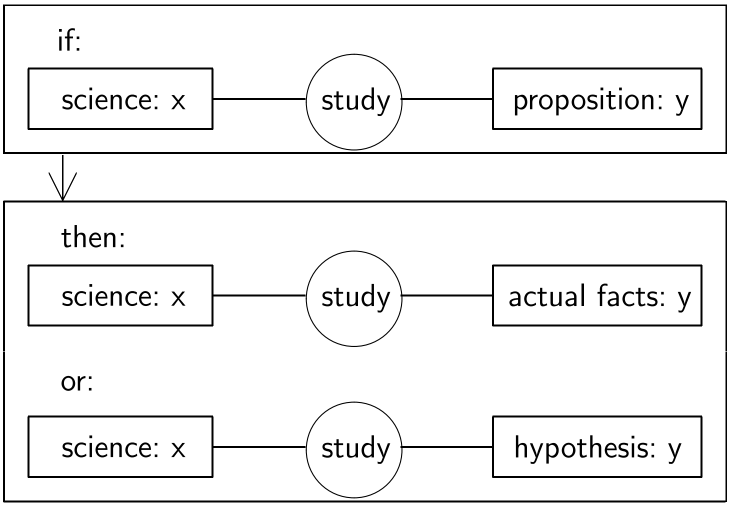

Fig. 4. Example of a clausal conceptual graph

A *clausal concept graph* of a power context family $\vec{\mathbb{K}}$ is defined as an existential concept graph $\mathfrak{G} := (V, E, \nu, \kappa, \rho)$ of $\vec{\mathbb{K}}$ over a variable set X with a designated pair $(p\mathfrak{G}, \{c_t\mathfrak{G} \mid t \in T\})$ consisting of a subgraph $p\mathfrak{G}$ of $\mathfrak{G}$ and a set $\{c_t\mathfrak{G} \mid t \in T\}$ of subgraphs of $\mathfrak{G}$ such that

1. $\mathfrak{G}$ is the union of $p\mathfrak{G}$ and all the $c_t\mathfrak{G}$ with $t \in T$, and
2. each X-interpretation admissible on $p\mathfrak{G}$ is also admissible on at least one $c_t\mathfrak{G}$ with $t \in T$.

$p\mathfrak{G} \to \bigvee_{t\in T} c_t\mathfrak{G}$ may be written instead of $\mathfrak{G}$; the subgraphs $p\mathfrak{G}$ and $c_t\mathfrak{G}$ $(t \in T)$ are called the *premise* and the *disjunctive conclusions*, resp. For subsets A and B of the attribute set M, $\bigwedge A \to \bigvee B$ is an *attribute clause* of $\mathbb{K}$ if $g \in A^I$ always implies gIm for at least one $m \in B$.

Proposition 3 *Let $\overline{\mathfrak{G}}$ be an existential concept graph of a power context family $\vec{\mathbb{K}}$ over a variable set X and let $\mathfrak{G}_s$ $(s \in S)$ and $\mathfrak{G}_t$ $(t \in T)$ be subgraphs of $\overline{\mathfrak{G}}$. Then $\bigwedge\{\mathfrak{G}_s \mid s \in S\} \to \bigvee\{\mathfrak{G}_t \mid t \in T\}$ is an attribute clause of the formal context $\mathbb{K}(X;\overline{\mathfrak{G}})$ if and only if $\bigcup_{s\in S} \mathfrak{G}_s \to \bigvee_{t\in T} \mathfrak{G}_t$ is a clausal concept graph of $\vec{\mathbb{K}}$ over X.*

Proposition 2 and 3 show that the theory of clausal concept graphs is essentially equivalent to the theory of attribute clauses of formal contexts. The advantage of this equivalence is that many results about attribute clauses can be transferred to clausal concept graphs which substantially enriches the research on Contextual Judgment Logic.

Corollary 2 $\mathfrak{G}_\emptyset \to \bigvee_{t\in T} \mathfrak{G}_t$ *is a clausal concept graph of* $\vec{\mathbb{K}}$ *over* X *if and only if for all* X*-interpretations* χ *into* G_0 *there exists a* $t_\chi \in T$ *such that* χ *is admissible on* $\mathfrak{G}_{t_\chi}$.

5 Generalizations of concept graphs

Concepts and concept graphs form a comprehensive core for a *semantics of Contextual Logic*. Although such a semantics offers a great variety of support for logical thinking, there is still the desire to use further conceptual structures. Here only two types of such structures shall be discussed briefly so that readers get at least an idea about the richness which still has to be explored.

As a first type of generalized graphs we want to consider *conceptual graphs with local negation*. For this, we start again with an example deduced from Peirce's statement cited at the beginning of section 2; the example is shown in Fig. 5 which presents an implicational conceptual graph with local negation. The diagram can

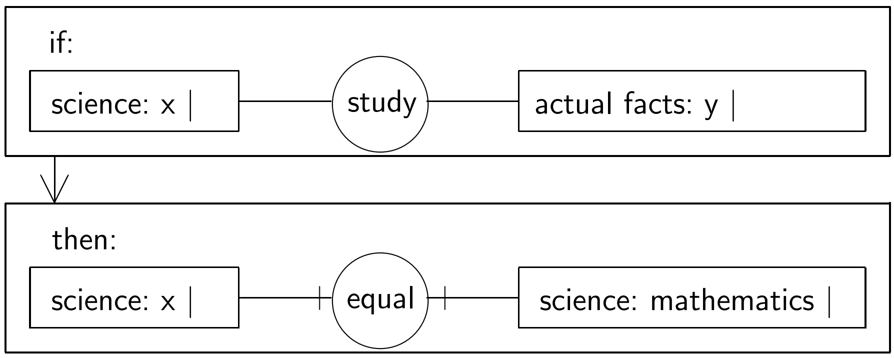

Fig. 5. Example of an implicational conceptual graph with local negation

be read: "If a science studies actual facts then this science is not mathematics" or "mathematics is a science which does not study actual facts". The additional vertical strokes which divide the spaces after the colon in the rectangular boxes allow to represent the negation of the object-concept-relation (see [Wi02]). The vertical strokes cutting the horizontal lines joining the rectangular boxes with the equal-circle indicate the negation of the relation "equal"; such localization means that "x is not mathematics".

Graphs with local negation have been introduced as *"protoconcept graphs"* in [Wi02]. An extensive elaboration of the logic system of those protoconcept graphs with their syntax and semantics can be found in [Kl05]. The logic system of *concept graphs with negation* (and its relationship to predicate logic) which can be understood as a mathematization of a large fragment of Sowa's theory of conceptual graphs, has been impressively worked out and published in the Springer Lecture Notes in Artificial Intelligence [Da03].

As a second type of generalized graphs we want to mention *conceptual graphs with a modal component*. Nested conceptual graphs may be understood to have a modal component. This becomes clear through a mathematical representation of nested conceptual graphs by *triadic concept graphs* which have been invented in [Wi98]. The discussion in that paper clarifies that one should consider not only nestings, but also subdivisions with overlappings. As an example for this, a triadic concept graph is shown which represents a diatonic modulation from C-major to A-major with its constitutive chord overlappings. Triadic concept graphs are based on a triadic power context family $\vec{\mathbb{K}} := (\mathbb{K}_0, \mathbb{K}_1, \mathbb{K}_2, \ldots)$ with $\mathbb{K}_k := (G_k, M_k, B, I_k)$ and $G_k \subseteq G_0^k$ $(k = 0, 1, 2, ...)$ where B is always a set of *modalities*. It can be shown that the triadic concept graphs of a triadic power context family always form a complete lattice with respect to the generalization order. It turns out that the generalization order may be differently defined, depending on the assumed background knowledge, respectively (cf. [GW00]).

A *concept graph with subdivision* is a mathematical structure derived from a triadic power context family. The aim of introducing concept graphs with subdivision is to represent modal information mathematically. This has been demonstrated in [SW03] by an example, namely by a comparison of the two famous paintings: the "Darmstädter Madonna" and the "Dresdner Madonna". Based on the notion of a *conceptual content*, the concept graphs with subdivision of a triadic power context family has been proved to form a complete lattice with respect to the *information order* (cf. [SW03]).

Finally, it shall be pointed out that a *conceptual theory and methodology of semantic structures*, named *"semantology"*, are under development from which the theory and practice of concept graphs and their generalizations, and therefore of Contextual Judgment Logic could benifit. The initial paper [GW06] discusses, from the view of Peirce's classification of sciences, a *three-fold semantics* of conceptual knowledge: the mathematical, the philosophical and an application-oriented semantics. Examples from *Formal Concept Analysis* are considered. The second paper [EW07] extends the discussion of the three-fold semantics to *Conceptual Knowledge Processing* by using the extensive analysis of methods in Formal Concept Analysis and Contextual Logic presented in [Wi06]. A special case-study about applications of semantology in music is offered in [WW07]. For understanding how mathematical methods can be applied in the real world, the relationship between mathematics and concept analysis is analysed in [Wi07], in particular by the three-fold semantics of concept analysis in Conceptual Knowledge Representation.

References

[Da03] F. Dau: The logic system of concept graphs with negation. Dissertation, TU Darmstadt 2003. Shaker Verlag, Aachen 2003.

[EW07] P. Eklund, R. Wille: Semantology as basis for Conceptual Knowledge Processing. In: S. O. Kuznetsov, St. Schmidt (eds.): *Formal Concept Analysis. ICFCA 2007.* LNAI **4390**. Springer, Heidelberg 2007, 18–38.

[Ga05] B. Ganter: Contextual attribute logic of many-valued contexts. In: B. Ganter, R. Wille, K. E. Wolff (eds.): *Formal Concept Analysis. State-of-the-Art Survey.* LNAI **3626**. Springer, Heidelberg 2005, 101-113.

[GW99a] B. Ganter, R. Wille: *Formal Concept Analysis: mathematical foundations.* Springer, Heidelberg 1999; German version: Springer, Heidelberg 1996.

[GW99b] B. Ganter, R. Wille: Contextual attribute logic. In: W. Tepfenhart, W. Cyre (eds.): *Conceptual structures: standards and practices.* LNAI **1640**. Springer, Heidelberg 1999, 377–388.

[GW06] P. Gehring, R. Wille: Semantology: basic methods for knowledge representations. In: H. Schärfe, Pascal Hitzler, Peter Øhrstrøm (eds.): *Conceptual structures: inspiration and application.* LNAI **4068**. Springer, Heidelberg 2006, 215–228.

[GW00] B. Groh, R. Wille: Lattices of triadic concept graphs. In: B. Ganter, G. Mineau (eds.): *Conceptual structures: logical, linguistic, and computational issues.* LNAI **1867**. Springer, Heidelberg 2000, 332-341.

[He74] H. von Hentig: Magier oder Magister? Über die Einheit der Wissenschaften im Verständigungsprozeß. Suhrkamp Taschenbuch **207**. Suhrkamp, Frankfurt 1974.

[Ka88] I. Kant: Logic. Dover, Mineola N.Y. 1988.

[Kl05] J. Klinger: The logic system of protoconcept graphs. Dissertation, TU Darmstadt 2005. Shaker Verlag, Aachen 2005.

[Pe92] Ch. S. Peirce: Reasoning and the logic of things. Edited by K. L. Ketner; with an introduction by K. L. Ketner and H. Putnam. Havard University Press, Cambridge 1992.

[Pr98] S. Prediger: Kontextuelle Urteilslogik mit Begriffsgraphen. Ein Beitrag zur Restrukturierung der mathematischen Logik. Dissertation. TU Darmstadt 1998. Shaker, Aachen 1998.

[SW03] L. Schoolmann, R. Wille: Concept graphs with subdivision: a semantic approach. In: A. de Moor, W. Lex, B. Ganter (eds.): *Conceptual structures for knowledge creation and communication.* LNAI **2746**. Springer, Heidelberg 2003, 271–281.

[So84] J. F. Sowa: Conceptual structures: Information processing in mind and machine. Adison-Wesley, Reading 1984.

[So92] J. F. Sowa: Conceptual graphs summary. In: T. E. Nagle, J. A. Nagle, L. L. Gerholz, P. W. Eklund (eds.): *Conceptual structures: current research and practice.* Ellis Horwood, 1992, 3–51.

[Wi82] R. Wille: Restructuring lattice theory: an approach based on hierarchies of concepts. In: I. Rival (ed.): *Ordered sets.* Reidel, Dordrecht-Boston 1982, 445–470.

[Wi96] R. Wille: Restructuring mathematical logic: an approach based on Peirce's pragmatism. In: A. Ursini, P. Agliano (eds.): *Logic and algebra.* Marcel Dekker, New York 1996, 267-281.

[Wi97] R. Wille: Conceptual Graphs and Formal Concept Analysis. In: D. Lukose, H. Delugach, M. Keeler, L. Searle, J. F. Sowa (eds.): *Conceptual Structures: Fulfilling Peirce's Dream.* LNAI **1257**. Springer, Heidelberg 1997, 290–303.

[Wi98] R. Wille: Triadic concept graphs. In: M. L. Mugnier, M. Chein (eds.): *Conceptual structures: theory, tools and applications.* LNAI **1453**. Springer, Heidelberg 1998, 194-208.

[Wi00] R. Wille: Boolean Concept Logic. In: B. Ganter, G. Mineau (eds.): *Conceptual structures: logical, linguistic, and computational issues.* LNAI **1867**. Springer, Heidelberg 2000, 317-331.

[Wi01] R. Wille: Boolean Judgment Logic. In: H. S. Delugach, G. Stumme (eds.): *Conceptual structures: broadening the base.* LNAI **2120**. Springer, Heidelberg 2001, 115-128.

[Wi02] R. Wille: Existential concept graphs of power context families. In: U. Priss, D. Corbett, G. Angelova (eds.): *Conceptual structures: integration and interfaces.* LNAI **2393**. Springer, Heidelberg 2002, 382–395.

[Wi03] R. Wille: Conceptual content as information - basics for Conceptual Judgment Logic. In: A. de Moor, W. Lex, B. Ganter (eds.): *Conceptual structures for knowledge creation and communication.* LNAI **2746**. Springer, Heidelberg 2003, 1–15.

[Wi04] R. Wille: Implicational concept graphs. In: K. E. Wolff, H. Pfeiffer, H. Delugach (eds.): *Conceptual structures at work.* LNAI **3127**. Springer, Heidelberg 2004, 52–61.

[Wi06] R. Wille: Methods of Conceptual Knowledge Processing. In: R. Missaoui, J. Schmid (eds.): *Formal Concept Analysis. ICFCA 2006.* LNAI **3874**. Springer, Heidelberg 2006, 1–29.

[Wi07] R. Wille: Formal Concept Analysis as applied lattice theory. In: R. Bělohlvek, S. Ben Yahia, E. Mephu Nguifo (eds.): *Concept lattices and their applications. CLA 2006.* LNAI . Springer, Heidelberg 2007, .

[WW07] R. Wille, R. Wille-Henning: Towards a semantology of music. In: U. Priss, S. Polovina (eds.): *Conceptual structures: knowledge architectures for smart applications.* LNAI . Springer, Heidelberg 2007, .