An FDP-Algorithm for Drawing Lattices

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Abstract. In this work we want to discuss an algorithm for drawing line diagrams of lattices based on force directed placement (FDP). This widely used technique in graph drawing introduces forces acting on nodes and lines. A balanced state of the system will result in a diagram fulfilling the desired properties due to the underlying physical model. In our framework the aim was to maximize the conflict distance. In contrast to existing programs our approach provides attribute additive diagrams since forces act on $\bigwedge$-irreducibles only. Another relevant aspect is a careful initialization that helps to minimize the number of edge crossings.

1 Motivation

We observe a growing demand on visualizations of concept lattices for representing knowledge in FCA. Several programs [1, 7, 15] use diagrams for exploring and analyzing database structures. Unfortunately, the automatic layout of the diagrams remains a difficult task. In order to gain the acceptance of the user, who is in general not an expert, the drawings should be easily readable. However, nobody knows exactly what that means since it is obviously not possible to mathematize human esthetic sensations. Nevertheless, there exist some algorithms for drawing lattices with the computer [1, 9, 19] based on different approaches. In this work we want to present such an attempt combining the FCA view on diagram layouting with tools developed in the theory of graph drawing.

2 Preliminaries

2.1 How to Draw a Diagram

The *graph drawing* community developed a variety of methods to classify layout algorithms [5]. First we have the *drawing conventions* declaring general constraints of the resulting drawings. In our attempt this includes the following:

1. *line diagrams*, a common principle for drawing lattices automatically,
2. *upward diagrams*, sometimes also called Hasse-diagram, a common method in order to avoid arrows on diagram lines,
3. *attribute additive diagrams*, a principle introduced more generally in [11].

While the first two conventions are canonical in lattice drawing algorithms, the third is not. Also *layer diagrams* [9] or *hybrid diagrams* [2] are employed. The first alternative seems less applicable for our purpose since in general it does not emphasize the grid structure of a lattice. The second however offers some advantages in case of non-distributive lattices[1]. Nevertheless it still adheres to the layer convention.

The second technique to create algorithms is the employment of *esthetic criteria*. They are mostly given by optimization tasks whose compliance is supposed to increase the diagrams quality, i.e. readability. Examples include

- minimize the number of slopes,
- maximize the smallest angle between incident lines,
- put the nodes onto an orthogonal grid.

There is little empirical analysis about the importance of that criteria. Two studies were made [16, 17] but only for general lattices. They highlighted the criterion of *minimizing the number of edge crossings*. Therefore we tried to emphasize it in our algorithm. The second criterion taken into our consideration is *maximizing the conflict distance* [12], i.e. the least distance between a node and a non-incident line.

Finally, diagram algorithms are distinguished in classes characterizing the way how the actual layout process is done. The *layer method* (see [5] for an overview) is fairly prominent [2] as well as *force directed methods* (introduced in [8], see [5] for an overview) implemented in [19] or a combination of both [9]. We decided to implement a force directed method since it is a natural way to maximize the conflict distance. Unlike other attempts, we keep the diagram attribute additive, thereby better satisfying the esthetic criterion of *displaying symmetries*.

2.2 Diagrams of Lattices

We consider diagrams in the usual way (see for instance [14] for a formal definition). Instead of the lattice $\underline{\mathfrak{V}} = (\mathfrak{V}, \leq)$ itself we draw only its graph $(\mathfrak{V}, \prec)$ (where $\prec$ denotes the upward neighbourhood relation in $\underline{\mathfrak{V}}$). A *line diagram* (briefly *diagram*) is an injective mapping pos fulfilling the upward drawing convention that assigns a point $\mathrm{pos}(v)$ in the Euclidian plane (called node) to each lattice element v. An element $e = (v, w)$ of $\prec$ is mapped to a straight line segment between $\mathrm{pos}(v)$ and $\mathrm{pos}(w)$, for convenience we write $\mathrm{pos}(e)$ for the image.

Next we want to remind the already mentioned attribute additivity [11, 18]. Since we consider lattices instead of concept lattices in this work, we sloppily define: A diagram of a lattice $\underline{\mathfrak{V}}$ is *attribute additive* if the $\bigwedge$-irreducibles $m \in M(\underline{\mathfrak{V}})$ are assigned to vectors $\mathrm{vec}(m)$ and all elements v are mapped to the sum of the vectors of $\bigwedge$-irreducibles greater or equal than them, i.e.

$$\mathrm{pos}(v) = \sum_{m \geq v} \mathrm{vec}(m).$$

[1] That can be easily observed at the lattice M_n. Hybrid diagrams do not push the 0-element of the lattice disproportional downwards.

Finally we introduce the *conflict distance* due to [12]:

Definition 1 *Let* $G = (V, E)$ *be a simple graph and* $\text{pos}(G)$ *a diagram of it. Let* $v \in V$ *be a vertex and* $e \in E$ *be a non-incident line w.r.t.* v*. The* node-line-distance *between* $\text{pos}(v)$ *and* $\text{pos}(e)$ *is the least Euclidian distance between* $\text{pos}(v)$ *and any point* $w \in \text{pos}(e)$*. The* conflict distance *of* $\text{pos}(G)$ *is the smallest of all node-line-distances in* $\text{pos}(G)$*.*

2.3 Left-Relations on Lattices

Left-relations give a possibility to characterize planar lattices and to describe plane diagrams of them [20–22]. They are closely related to *conjugate orders* [6]. Intuitively, a left-relation on a diagram describes, whether a lattice element v is left or right of another element w incomparable to v. In contrary, comparable elements are considered to be above or below each other. We do not need a formal definiton here but only one result given in [21] which explains the heuristics we use in Section 3.1, namely the *first planarity condition (FPC)*. In Figure 1 an intuitive explanation for the necessity of the FPC is given.

Definition 2 *[21] A conjugate relation*[2] R *on a lattice* $\underline{\mathfrak{V}}$ *fulfills the* first planarity condition (FPC) *if the implication* $m_i \, R \, m_k \, R \, m_j \implies m_k > (m_i \wedge m_j)$ *holds for all* $\bigwedge$*-irreducibles* $m_i, m_k, m_j \in M(\underline{\mathfrak{V}})$*.*

Proposition 1 *[21] Let* L *be a left-relation on a lattice* $\underline{\mathfrak{V}}$*, then the following equivalence holds:*

$$L \text{ satisfies the FPC} \iff \underline{\mathfrak{V}} \text{ is planar.}$$

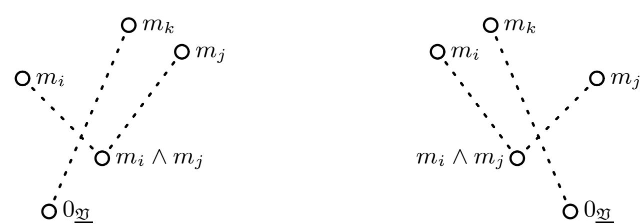

Fig. 1. When considering a diagram of a lattice, the necessity of the FPC is obvious for its planarity: If $m_i \, L \, m_k \, L \, m_j$ or $m_j \, L \, m_k \, L \, m_i$ holds then also $m_k > (m_i \wedge m_j)$. Otherwise every chain of diagram edges from m_k to the bottom element of the lattice intersects with a chain of edges from either m_i or m_j to $m_i \wedge m_j$.

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[2] That is a relation satisfying $R \cup R^{-1} = \|$, where $\|$ denotes the incomparability relation in $\underline{\mathfrak{V}}$.

3 The FDP-Algorithm

Force Directed Placement (FDP) is a widely used technique for drawing diagrams of graphs [8, 10, 13] and is also applied for the layout of lattice diagrams [19, 9]. Nodes and sometimes also edges are considered as physical bodies which interact by a set of repulsive and attractive forces. If the system is in a balanced state, i.e. a local minimum of the appropriate energies, then the resulting diagram will hopefully be nicely drawn w.r.t. the properties that determine the forces. In general an FDP-algorithm consists of three parts:

– choice of an adequate model,
– determination of an initial state,
– seeking for a balanced state using an optimization algorithm.

In many cases attention is rather put on the optimization process, for instance with simulated annealing or genetic algorithms. The models are kept simple and the initial state is rather arbitrary. The two layout styles provided by [19] distribute the lattice elements on layers. This may work well if either a global minimum of the included energies is found or if several minima are found and ranked by a quality function acting on the respective diagrams.

Our approach tries to include a more sophisticated initialization. This is done by applying a heuristics to minimize the number κ of edge crossings in the diagram. The final optimization step is not allowed to shift any node out of its surrounding cell. Therefore, κ does not change as well as the left-relation of the diagram.

3.1 Initialization

As mentioned already this is the crucial step of our approach. Since the diagram is determined just by the coordinates of the $\bigwedge$-irreducibles we only have to assign vectors to them. We distinguish between the coatoms and other $\bigwedge$-irreducible elements. The first are distributed on a parable

$$y = -0.09x^2 - 1.75$$

which was derived heuristically from the position of the $\bigwedge$-irreducibles in suitable diagrams of the boolean lattices B_4 and B_5. In case of an even number of coatoms they obtain the coordinates given by the x-component $\pm 0.9, \pm 2.7, \ldots$, if otherwise an odd number occurs, we assign $0, \pm 1.8, \pm 3.6, \ldots$. All other vectors are obtained by

$$\mathrm{pos}(m_i) = \Delta_i + \sum_{m_j > m_i} \mathrm{pos}(m_j),$$

i.e. the mean of the vectors of $\bigwedge$-irreducibles m_j situated above m_i. If the $\bigwedge$-irreducible elements above m_i form a chain then the respective coordinates will lay on a line in the initial diagram. The symbol Δ_i represents a small shift which is necessary when two $\bigwedge$-irreducibles share the same upper neighbour.[3]

[3] Otherwise they would obtain the same diagram vector.

Finally we have to clarify in which way the nodes of the $\bigwedge$-irreducibles are sorted. We mentioned already that we want to minimize the number of edge crossings in the diagram. Since there exists no efficient analytical algorithm for that purpose we introduce a heuristical model called *planarity enhancer*.

The underlying idea is the following: *Similar concepts, i.e. concepts sharing similar intents should be positioned more closely than non-similar ones*. This is motivated by the FPC, see Proposition 1. It refers to the fact that, in a plane diagram, a $\bigwedge$-irreducible n should be drawn "inbetween" m_1 and m_2 only if it is greater than the infimum $m_1 \wedge m_2$, i.e. if $m_1 \wedge n \geq m_1 \wedge m_2$ or $m_2 \wedge n \geq m_1 \wedge m_2$. Hence pairs (m, n) of elements with a large infimum, i.e. with a small cardinality of $M(m \wedge n)$[4] shall be drawn close together. In order not to favor those $\bigwedge$-irreducible elements situated near the bottom of the lattice we only count $\bigwedge$-irreducibles not included in $M(m \vee n)$. Based on these ideas, we define the *sup-inf-distance* between two incomparable $\bigwedge$-irreducibles m and n as follows:

$$d_{SI}(m, n) := |M(m \wedge n) \setminus M(m \vee n)| - 1.$$

This allows us to create a complete weighted graph Γ_{SI} where $M(\underline{\mathfrak{V}})$ is the vertex set and each edge between m and n obtains the weight $d_{SI}(m, n)$. This graph can be considered as a 2-dimensional physical body with rings instead of nodes and springs instead of edges. The spring force is due to the standard physical model given as $F = -k \cdot x$, where k is the spring constant and x the displacement of the idle state. The springs are thought to be in rest position if their length is equal to the weight in the appropriate graph. This results in a system energy

$$E_{SI} = \sum_{m_i, m_j \in M(\underline{\mathfrak{V}})} (|\mathrm{pos}(m_i) - \mathrm{pos}(m_j)| - d_{SI}(m_i, m_j))^2.$$

We find the force acting on a vector $\mathrm{pos}(m)$ by differentiating the last formula to each $\bigwedge$-irreducible m yielding

$$F_{SI}(\mathrm{pos}(m)) = -2 \cdot \sum_{n \in M} \frac{|\mathrm{pos}(m) - \mathrm{pos}(n)| - d_{SI}(m, n)}{|\mathrm{pos}(m) - \mathrm{pos}(n)|} \cdot (\mathrm{pos}(m) - \mathrm{pos}(n)).$$

After reaching an equilibrium state of this system by applying a robust minimizer we do a linear regression of the emerged scatter plot, followed by an orthogonal projection of the nodes onto the obtained line. The sorting of the nodes representing the attributes finally gives their sorting relation [20]. This is a relation indicating in which order the $\bigwedge$-irreducibles with common upper neighbour shall be sorted from left to right.

[4] With $M(v)$ we denote the set of $\bigwedge$-irreducibles greater or equal than v.

3.2 The Model

Implemented Forces

Since we want to draw attribute additive diagrams the energies and forces do not act on the node of an element $v \in \mathfrak{V}$ itself but on the vectors $\mathrm{vec}(m)$ of the $\bigwedge$-irreducibles $m \in M(v)$. Therefore, the systems energy is the sum of the ones inherent in the vectors of the elements of $M(\underline{\mathfrak{V}})$. Equivalently, the resulting force is a vector of forces on vectors on $\bigwedge$-irreducibles (see Figure 2). This yields

$$E = \sum_{m \in M} E(\mathrm{vec}(m_i)) \text{ and } F = -\nabla E = (F(\mathrm{vec}(m_1)), \ldots F(\mathrm{vec}(m_0)))^T.$$

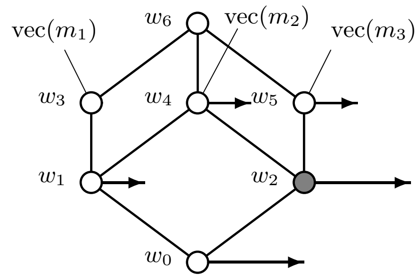


Fig. 2. The force acting on the node w_2 affects the nodes w_0, w_1, w_4 and w_5 too. Since the latter three contain only one of the $\bigwedge$-irreducibles m_2 or m_3, the resulting force is half the original.

The aim of the model is to maximize the conflict distance. It is therefore based on a *repulsive force* F_{rep}. Since we want to avoid the occurence of any conflict, the following definition on the graph $(\mathfrak{V}, \prec)$ of a lattice is obvious:

$$F_{rep} = -\nabla E_{rep}, \quad E_{rep} = \sum_{v \in V} \sum_{e \in \prec, v \notin e} \frac{1}{d(\mathrm{pos}(v), \mathrm{pos}(e))}.$$

Thereby $d(\mathrm{pos}(v), \mathrm{pos}(e))$ is the distance between a node and an edge introduced in Definition 1. A node positioned on a non-incident line causes E_{rep} to be infinite, hence this never results in a stable state.

To prevent the diagram from blowing up we need an *attractive force* F_{att} which minimizes the edge length. This is done due to the physical model of a spring supplying the formulas

$$F_{att} = -\nabla E_{att}, \quad E_{att} = \sum_{e \in \prec} |\mathrm{pos}(e)|^2.$$

Finally, we employ a *gravitative force* F_{grav} to ensure the upward-drawing-constraint. It acts on vectors $n_i = \mathrm{vec}(m_i)$ of $\bigwedge$-irreducibles only. Since this force is supposed to be dependent on the angle $\varphi(n)$ between the vector of an $\bigwedge$-irreducible and a horizontal line, we define as follows:

$$F_{grav}(n_i) = -\frac{\mathrm{d}E_{grav}}{\mathrm{d}n_i},$$

$$\frac{\mathrm{d}E_{grav}(n_i)}{\mathrm{d}\varphi(n_i)} = \frac{\sin^2\varphi(n_i) - \sin^2\varphi_0}{\sin^2\varphi(n_i)} \cdot \begin{cases} 1 \,, \varphi(n_i) \in [0, \varphi_0] \\ 0 \,, \varphi(n_i) \in [\varphi_0, \pi - \varphi_0] \\ -1 \,, \varphi(n_i) \in [\pi - \varphi_0, \pi] \end{cases}$$

This formula seems to be quite clumsy. However, the underlying idea is to push "nearly horizontal" vectors stronger downwards than the more slanted ones (see Figure 3). The gravitative force may not act if the vectors are "vertical enough", i.e. if their angle is between φ_0 and $\pi - \varphi_0$, which is chosen by (see Figure 3 for an explanation)

$$\varphi_0 := \frac{\pi}{|M(\underline{\mathfrak{B}}(\mathbb{K}))| + 1} \,.$$

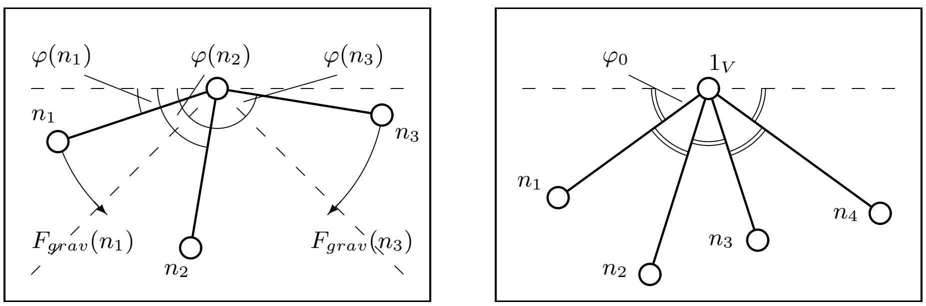


Fig. 3. left: The diagonal lines represent φ_0 and $\pi - \varphi_0$. The nodes n_1 and n_3 are pushed down by the gravitative force, but not n_2.
right: If all attribute concepts are coatoms then their vectors can be assigned, s.t. the angles between two of them and to the horizontal dashed line are all equal to φ_0.

Integrating by φ_0 in consideration of the reasonable boundary condition given by $\varphi(n_i) \in \{\varphi_0, \pi - \varphi_0\} \implies E(n_i) = 0$ to make the energy function continuous in $(0, \pi)$ yields

$$E(n_i) = \begin{cases} \varphi(n_i) + \cot\varphi(n_i)\sin^2\varphi_0 + E_0 \,, 0 \le \varphi(n_i) \le \varphi_0 \\ -\varphi(n_i) - \cot\varphi(n_i)\sin^2\varphi_0 + E_1 \,, \pi - \varphi_0 \le \varphi(n_i) \le \pi \end{cases}$$
$$\text{with } E_0 = E_1 - \pi = -\varphi_0 - \sin\varphi_0\cos\varphi_0 .$$

The total energy and the total force respectively are obtained as a linear combination of its components, i.e.

$$E = r \cdot E_{rep} + a \cdot E_{att} + g \cdot E_{grav},$$
$$F = r \cdot F_{rep} + a \cdot F_{att} + g \cdot F_{grav}.$$

Calculation of the Forces

The calculation of the repulsive force demands the observation of various cases. Firstly we have to distinguish how a node and a non-incident line are related to each other in the plane. We discover the three possibilities given in Figure 4.

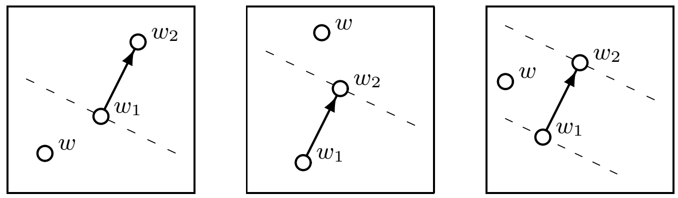


Fig. 4. The three cases of node-edge-relationship.

Secondly we must take into consideration which of the sets $M(v)$, $M(v_1)$ and $M(v_2)$ contain the $\bigwedge$-irreducible m. The alternatives are depicted in Figure 5.

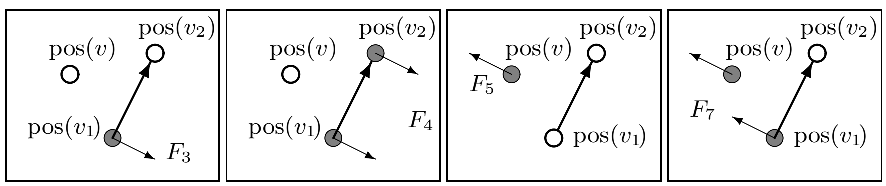


Fig. 5. Four possibilities of forces acting between a node and a non-incident line depending on the containment of a $\bigwedge$-irreducible m in the sets $M(v)$, $M(v_1)$ and $M(v_2)$ of the respective lattice elements (shaded).

After some calculation we find the following table determining the repulsive force for each case. Due to abbreviation reasons we write n instead of pos(m) for attribute vectors, w instead of pos(v) for nodes and f instead of pos(e) for vectors of diagram lines. The symbol e_w denotes the unit vector of a node w, furthermore $n_+(z)$ is the vector arising from the vector z by turning by $\pi/2$ in positive direction of rotation and l either $+1$ in case of the node w being situated left of the line f and -1 otherwise.

	F_3	F_4	F_5	F_7
case 1	$e_{(w_1-w)}$	$e_{(w_1-w)}$	$e_{(w-w_1)}$	0
case 2	0	$e_{(w_2-w)}$	$e_{(w-w_2)}$	$e_{(w-w_1)}$
case 3	$-\sqrt{\frac{(w_2-w)^2-\lvert h\rvert^2}{\lvert f\rvert^2}}\cdot\frac{n_+(f)\cdot l}{\lvert f\rvert}$	$-\frac{n_+(f)\cdot l}{\lvert f\rvert}$	$\frac{n_+(f)\cdot l}{\lvert f\rvert}$	$\sqrt{\frac{(w_1-w)^2-\lvert h\rvert^2}{\lvert f\rvert^2}}\cdot\frac{n_+(f)\cdot l}{\lvert f\rvert}$

Table 1. Summary of all occuring forces $\frac{\mathrm{d}d(w,f)}{\mathrm{d}n_i}$ for the different cases.

The appropriate repulsive force acting on an attribute vector n_i is given by

$$F_{rep}(n_i) = -\frac{\mathrm{d}E_{rep}(n_i)}{\mathrm{d}n_i} = -\sum_{v\in V}\sum_{e\in\prec, v\notin e} \frac{\mathrm{d}\left(\frac{1}{d(w,f)}\right)}{\mathrm{d}n_i}$$
$$= \sum_{v\in V}\sum_{e\in\prec, v\notin e} \frac{1}{d(w,f)^2}\cdot\frac{\mathrm{d}d(w,f)}{\mathrm{d}n_i}.$$

For the attractive force we gain the formula

$$F_{att}(n_i) = -\sum_{e\in\prec}\frac{\mathrm{d}|f|^2}{\mathrm{d}n_i} = \quad 2\cdot\sum_{\substack{v_1 v_2\,\in\prec,\\ m_i\,\in\, v_1\setminus v_2}} f,$$

and finally for the gravitative force

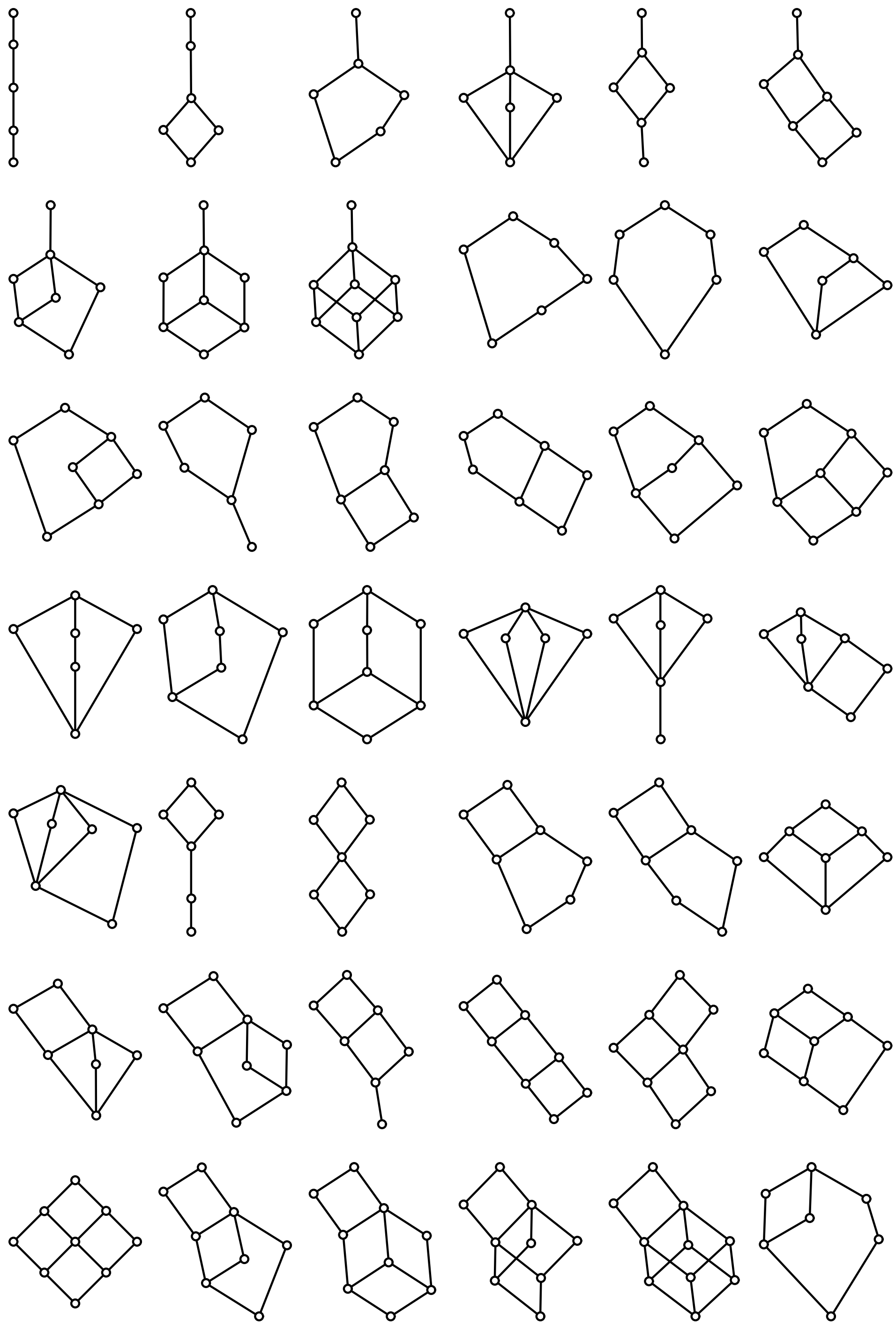
$$F_{grav}(n_i) = -\frac{\mathrm{d}E_{grav}(n_i)}{\mathrm{d}n_i} = n_+(n_i)\cdot\frac{\sin^2\phi(n_i) - \sin^2\phi_0}{y(n_i)^2}\cdot\begin{cases} 1\ , 0 < \phi(n_i) < \phi_0 \\ -1\ , \phi_0 < \phi(n_i) < \pi \end{cases}.$$

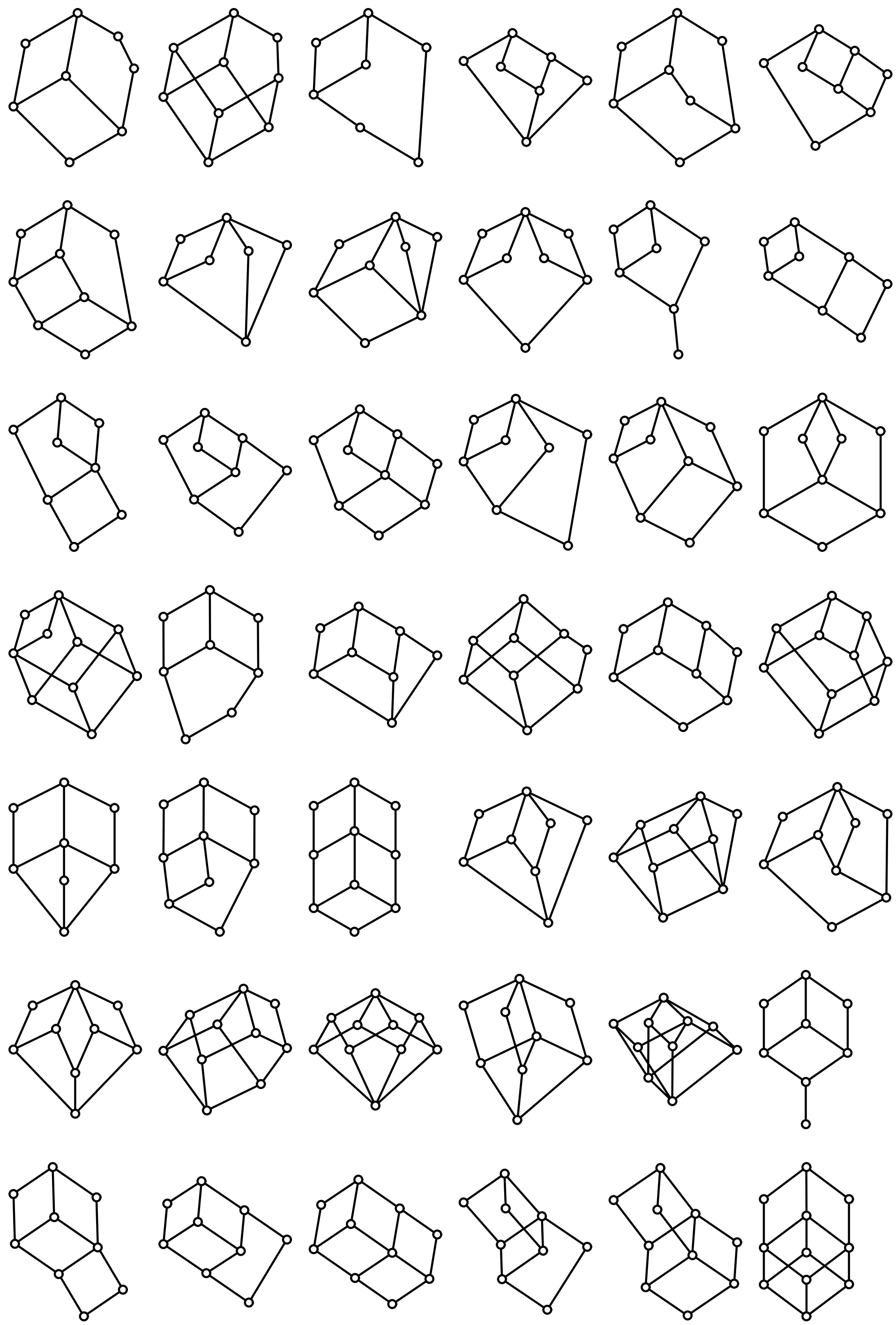
3.3 The Minimizing Algorithm

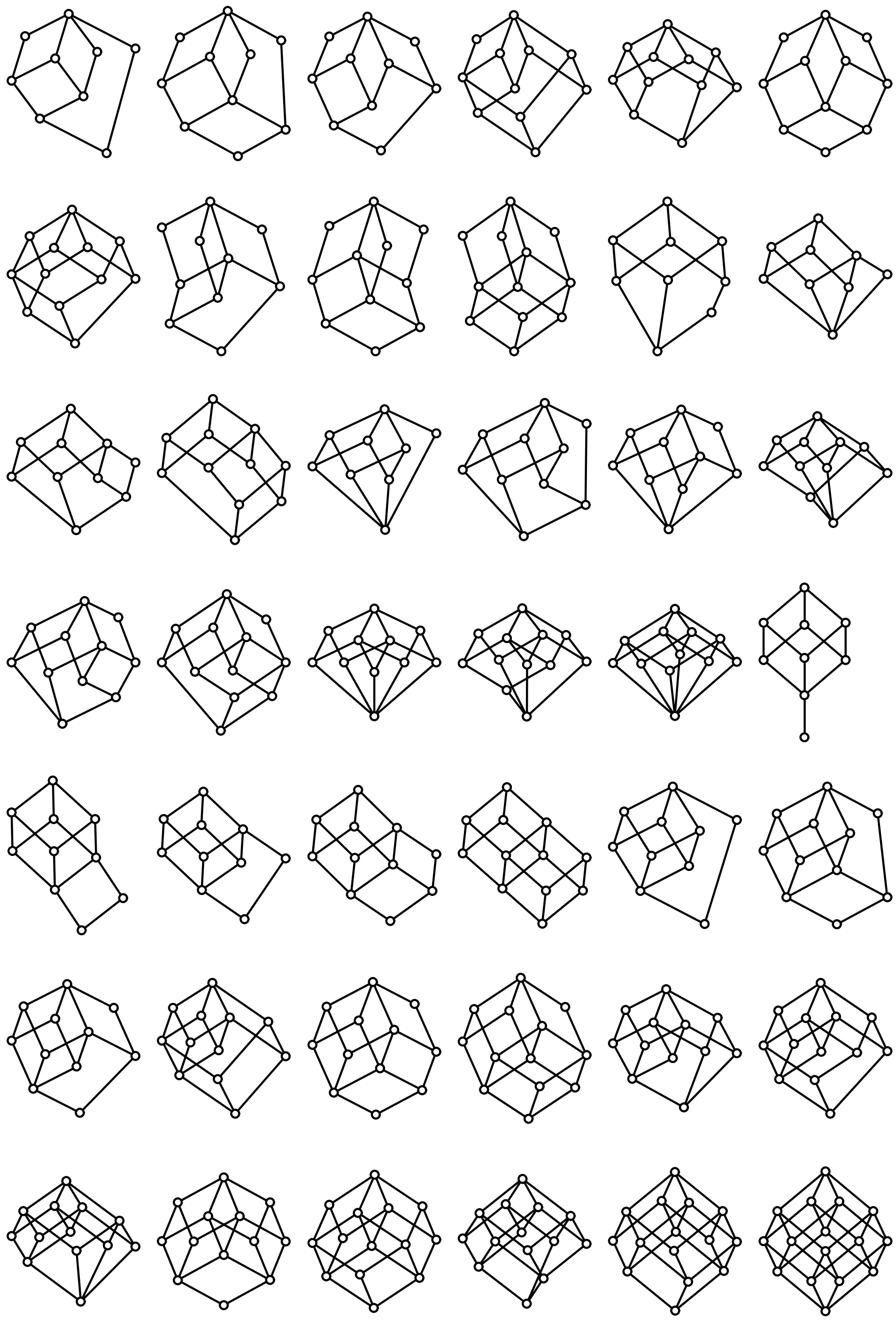
This step requires robustness rather than speed. We definitely want to avoid a node stepping out of its cell in order to keep the shape of the initialized diagram. Running time is no issue, the calculation of the local minimum is done in less than a second in most cases. We chose the well known conjugate gradient method as optimization algorithm (see for instance [4]).

4 Results

Instead of displaying some arbitrary chosen diagrams we just give the drawings of all lattices with four $\bigwedge$-irreducible elements. The calculation of all diagrams took less than 10 seconds. We also produced the respective diagrams of lattices with five $\bigwedge$-irreducible elements. The process of determining all 13596 lattices and drawing their diagrams took 25 minutes on an ordinary PC, which is an average of about 0.1 seconds per diagram. Unfortunately they are not publishable due to space limitations.







5 Conclusion

We presented an algorithm for drawing lattice diagrams designed in three steps. The *initialization* serves for implementing some esthetic criteria whose compliance is desired for the resulting drawing. In our particular approach we chose *minimization of the number of edge crossings* as such an criterion. The second step, namely the choice of a *model*, aims to *maximize the conflict distance*, thereby considering particular *drawing conventions*. Here we chose the *attribute additivity*. Finally, the optimization step seeks for a minimal energy state (i.e. a maximal conflict distance) while keeping the properties of the initial state and the model. Ino ur algorithm the underlying left-relation and the attribute additivity are preserved.

We think that our approach is fairly convenient for drawing lattice diagrams. The framework of an initialization employing esthetic criteria and an FDP-algorithm including drawing conventions gives a clear image of the desired properties of the resulting diagram. This modularity seems to be the main advantage of this approach.

As already mentioned, diagrams with few edge crossings are favoured by users. Therefore a drawing algorithm should emphasize that issue. Although our approach contains a very simple initialization heuristics only, all planar lattices with four $\bigwedge$-irreducibles are indeed drawn without edge crossings.

Layer assignment is a favoured way of lattice drawing algorithms. Even though we think that layer diagrams are satisfactory and user preferred for many (in particular distributive) lattices, one should not restrict to this convention in general.

Despite these advantages of our approach we do not think that the results are superior to those of other algorithms. In the following we want to give some possibilities that could improve the diagrams quality:

- The involvement of additional or different constraints like the *visualization of chains* (by chain decomposition) or the *hybrid convention* could result in more symmetrical diagrams.
- The initialization step can be improved by including techniques given in [20–22] to improve the quality of layouts of planar lattices. Also, we recently try to find strategies to characterize and draw “nearly planar” lattices as well.
- It may be useful to produce several diagrams that can be compared by some set of quality functions (proposed for instance in [3]).

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