Uniform Interpolation in General $\mathcal{EL}$ Terminologies

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Abstract. Recently, different forgetting approaches for knowledge bases expressed in different logics were proposed. For $\mathcal{EL}$ terminologies containing each atomic concept at most once on the left-hand side, an approach has been proposed based on sufficient, but not necessary acyclicity conditions. In this paper, we propose an approach for computing a uniform interpolant for general $\mathcal{EL}$ terminologies. We first show that a uniform interpolant of any $\mathcal{EL}$ terminology w.r.t. any signature always exists in $\mathcal{EL}$ enriched with least and greatest fixpoint constructors and show how it can be computed by reducing the problem to the computation of Most General Subconcepts and Most Specific Superconcepts for atomic concepts. Then, we give the exact conditions for the existence of a uniform interpolant in $\mathcal{EL}$ and show how it can be obtained using our algorithms.

1 Introduction

The importance of non-standard reasoning services supporting knowledge engineers in modelling a particular domain or in understanding existing models by visualizing implicit dependencies between concepts and roles was pointed out by the research community [3], [5]. An example of such reasoning services supporting knowledge engineers in different activities is the uniform interpolation. In particular for the understanding and the development of complex knowledge bases, e.g., those consisting of *general concept inclusions* (GCIs), the appropriate tool support would be beneficial. However, the existing approach [7] to uniform interpolation in $\mathcal{EL}$ is restricted to terminologies containing each atomic concept at most once on the left-hand side of concept inclusions and additionally satisfying sufficient, but not necessary acyclicity conditions. Lutz et al.[10] propose an approach to uniform interpolation in expressive description logics such as $\mathcal{ALC}$ w.r.t. general terminologies by encoding $\mathcal{ALC}$ TBoxes as concepts, which, however does not solve the problem of uniform interpolation in $\mathcal{EL}$.

Clearly, the existence of the results for such reasoning problems is closely related to the notion of fixpoint semantics. For instance, Baader [2] shows that the structurally similar problems of computing Least Common Subsumer and Most Specific Concept can always be solved in cyclic classical TBoxes w.r.t. to greatest fixpoint semantics. Similar results were obtained for general $\mathcal{EL}$ TBoxes with descriptive semantics[9] , however extended with the greatest fixpoint constructor ($\mathcal{EL}_\nu$). In this paper, we extend the above results by showing that uniform interpolants preserving all $\mathcal{EL}$ consequences of general $\mathcal{EL}$ terminologies w.r.t. an arbitrary signature can always be expressed in an extension of $\mathcal{EL}$ with least fixpoint and greatest fixpoint constructors μ, ν as well as the disjunction used only on the left-hand side of concept inclusions. We extend the previous approach [7] and propose the algorithms for computing such uniform interpolants for general $\mathcal{EL}$ terminologies based on the notion of *most general subconcepts* and *most specific superconcepts*.

In the usual application scenarios it is rather useful to obtain uniform interpolants expressed in the DL of the original terminology instead of introducing additional language constructs. Therefore, in addition to the above algorithms, we derive the existence criteria for uniform interpolants in $\mathcal{EL}$ (i.e., expressed without the above extension) and show how such a uniform interpolant can be obtained using our algorithms. Full proofs are available in the extended version of this paper [11].

2 Preliminaries

Let N_C and N_R be countably infinite and mutually disjoint sets of concept symbols and role symbols. An $\mathcal{EL}$ concept C is defined as

$$C ::= A|\top|C \sqcap C|\exists r.C$$

where A and r range over N_C and N_R, respectively. In the following, we use symbols A, B to denote atomic concepts and C, D to denote arbitrary concepts. A *terminology* or *TBox* consists of *concept inclusion* axioms $C \sqsubseteq D$ and *concept equivalence* axioms $C \equiv D$ used as a shorthand for $C \sqsubseteq D$ and $D \sqsubseteq C$. While knowledge bases in general can also include a specification of individuals with the corresponding concept and role assertions (ABox), in this paper we abstract from ABoxes and concentrate on TBoxes. The signature of an $\mathcal{EL}$ concept C or an axiom α, denoted by $\mathrm{sig}(C)$ or $\mathrm{sig}(\alpha)$, respectively, is the set of concepts and role symbols occurring in it. The signature of a TBox $\mathcal{T}$, in symbols $\mathrm{sig}(\mathcal{T})$, is analogously $N_C \cup N_R$. In what follows, we denote the set $N_C \cup \{\top\}$ as N_C^+.

Before introducing the fixpoint operators, we recall the semantics of the above introduced DL constructs, which is defined by the means of interpretations. An interpretation $\mathcal{I}$ is given by the domain $\Delta^\mathcal{I}$ and a function $\cdot^\mathcal{I}$ assigning each concept $A \in N_C$ a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and each role $r \in N_R$ a subset $r^\mathcal{I}$ of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. The interpretation of $\top$ is fixed to $\Delta^\mathcal{I}$. The interpretation of an arbitrary $\mathcal{EL}$ concept is defined inductively, i.e., $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$ and $(\exists r.C)^\mathcal{I} = \{x \mid (x, y) \in r^\mathcal{I}, y \in C^\mathcal{I}\}$. An interpretation $\mathcal{I}$ satisfies an axiom $C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$. $\mathcal{I}$ is a model of a TBox, if it satisfies all of its axioms. We say that a TBox $\mathcal{T}$ entails an axiom α, if α is satisfied by all models of $\mathcal{T}$. In combination with fixpoint constructors, we will additionally use *concept disjunction* $C \sqcup D$, the semantics of which is defined by $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$.

We now introduce the logics $\mathcal{EL}_{\mu(\sqcup),\nu}$, a fragment of monadic second order logics that we use to compute uniform interpolants of general $\mathcal{EL}$ TBoxes. $\mathcal{EL}_{\mu(\sqcup),\nu}$ is an extension of $\mathcal{EL}$ by the two fixpoint constructors, $\nu X.C_\nu$ [9] and $\mu X.C_\mu$ [4]. X is an element of the countably infinite set of concept variables N_V and C_ν, C_μ are constructed as follows:

$$C_\nu ::= X|A|\top|\nu X.C_\nu|C_\nu \sqcap C_\nu|\exists r.C_\nu$$

$$C_\mu ::= X|A|\top|\mu X.C_\mu|C_\mu \sqcup C_\mu|C_\mu \sqcap C_\mu|\exists r.C_\mu$$

where A ranges over atomic concepts and X ranges over N_V. All $\mathcal{EL}_\nu$ concepts and all $\mathcal{EL}_{\mu(\sqcup)}$ concepts are closed C_ν and C_μ expression, i.e., all concept variables are bound by the corresponding fixpoint constructor. Note that we define $\mathcal{EL}_\nu$ concepts and all $\mathcal{EL}_{\mu(\sqcup)}$ concepts in such a way, that no concept can contain both fixpoint constructors, i.e., we do not combine the two constructors within concepts. The semantics of the fixpoint constructors is defined using a mapping ϑ of concept variables to subsets of $\Delta^{\mathcal{I}}$. For an $\mathcal{EL}_{\mu(\sqcup),\nu}$ concept C and $W \subseteq \Delta^{\mathcal{I}}$, we denote a replacement of X by W as $C^{\mathcal{I},\vartheta[X\to W]}$. The semantics of $\mathcal{EL}_{\mu(\sqcup),\nu}$ concepts is defined by

$$(\nu X.C)^{\mathcal{I},\vartheta} = \bigcup\{W \subseteq \Delta^{\mathcal{I}} | W \subseteq C^{\mathcal{I},\vartheta[X\to W]}\}$$

$$(\mu X.C)^{\mathcal{I},\vartheta} = \bigcap\{W \subseteq \Delta^{\mathcal{I}} | C^{\mathcal{I},\vartheta[X\to W]} \subseteq W\}.$$

In order to allow for more succinct concept expressions, we use an extended version of the fixpoint constructs allowing for mutual recursion [12], [9]. The extended constructors have the form $\nu_i X_1...X_n.C_{\nu,1},...,C_{\nu,n}$ and $\mu_i X_1...X_n.C_{\mu,1},...,C_{\mu,n}$ with $1 \leq i \leq n$. The semantics is defined as follows: $(\nu_i X_1...X_n.C_1,...,C_n)^{\mathcal{I},\vartheta} = \bigcup\{W_i\}$ and $(\mu_i X_1...X_n.C_1,...,C_n)^{\mathcal{I},\vartheta} = \bigcap\{W_i\}$ such that there are $W_1,...,W_{i-1},W_{i+1},...,W_n$ with respectively $W_j \subseteq C_j^{\mathcal{I},\vartheta[X_1\to W_1,...,X_n\to W_n]}$ and $C_j^{\mathcal{I},\vartheta[X_1\to W_1,...,X_n\to W_n]} \subseteq W_j$ for $1 \leq j \leq n$.

3 TBox Inseparability and Uniform Interpolation

Intuitively, two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are inseparable w.r.t. a signature Σ if they have the same Σ consequences, i.e., consequences whose signature is a subset of Σ. Depending on the particular application requirements, the expressivity of those Σ consequences can vary from subsumption queries and instance queries to conjunctive queries. In this paper, we investigate forgetting based on concept inseparability of general $\mathcal{EL}$ terminologies defined analogously to previous work on inseparability, e.g., [8] or [7], as follows:

Definition 1. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two general $\mathcal{EL}$ TBoxes and Σ a signature. $\mathcal{T}_1$ and $\mathcal{T}_2$ are concept-inseparable w.r.t. Σ, in symbols $\mathcal{T}_1 \equiv^c_\Sigma \mathcal{T}_2$, if for all $\mathcal{EL}$ concepts C, D with $sig(C) \cup sig(D) \subseteq \Sigma$ holds $\mathcal{T}_1 \models C \sqsubseteq D$, iff $\mathcal{T}_2 \models C \sqsubseteq D$.*

Given a signature Σ and a TBox $\mathcal{T}$, the aim of uniform interpolation or forgetting is to determine a TBox $\mathcal{T}'$ with $\text{sig}(\mathcal{T}') \subseteq \Sigma$ such that $\mathcal{T} \equiv^c_\Sigma \mathcal{T}'$. $\mathcal{T}'$ is also called a *Uniform Interpolant (UI)* of $\mathcal{T}$ w.r.t. Σ. We call $\overline{\Sigma} = \text{sig}(\mathcal{T}) \setminus \Sigma$ the *forgotten* signature. In practise, the uniform interpolants are required to be finite. Therefore, in this paper, we investigate the existence of such uniform interpolants in $\mathcal{EL}$, i.e., uniform interpolants expressible by a finite set of finite axioms using only the language constructs of $\mathcal{EL}$. As demonstrated by the following example, in the presence of cyclic concept inclusions, a finite UI in $\mathcal{EL}$ might not exist for a particular $\mathcal{T}$ and a particular Σ.

Example 1. Forgetting the concept A in the TBox $\mathcal{T} = \{A' \sqsubseteq A, A \sqsubseteq A'', A \sqsubseteq \exists r.A, \exists s.A \sqsubseteq A\}$ results in an infinite chain of consequences $A' \sqsubseteq \exists r.\exists r.\exists r....A''$ and $\exists s.\exists s.\exists s....A' \sqsubseteq A''$ containing nested existential quantifiers of unbounded maximal depth.

Such infinite chain of consequences can be easily expressed in a finite way using the greatest fixpoint constructor ν and the least fixpoint constructor μ, thereby resulting in the inclusion axioms $A' \sqsubseteq \nu X.(A'' \sqcap \exists r.X)$ and $\mu X.(A' \sqcup \exists s.X) \sqsubseteq A''$. In the following, we show that for any $\mathcal{EL}$ TBox $\mathcal{T}$ and any signature Σ, a corresponding UI of $\mathcal{T}$ w.r.t. Σ in $\mathcal{EL}_{\mu(\sqcup),\nu}$ can always be computed. For this purpose, we reduce the problem of computing UI to the problem of computing *most general subconcepts* $\mathtt{MGS}(\Sigma, \mathcal{T}, A)$ and *most specific superconcepts* $\mathtt{MSS}(\Sigma, \mathcal{T}, A)$ for each concept $A \in \mathrm{sig}(\mathcal{T})$.

Definition 2. *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and Σ a signature. Further, let $A \in N_C$. A set of $\mathcal{EL}$ concepts C_i for $1 \leq i \leq n$ is $\mathtt{MSS}(\mathcal{T}, \Sigma, A)$, if:*

- *$sig(C_i) \subseteq \Sigma$ for all i,*
- *$\mathcal{T} \models \bigsqcup C_i \sqsubseteq A$ and $\mathcal{T} \not\models A \sqsubseteq \bigsqcup C_i$,*
- *for all $\mathcal{EL}$ concepts D with $sig(D) \subseteq \Sigma$ holds:*
  $\mathcal{T} \models D \sqsubseteq A$, iff $\mathcal{T} \models D \sqsubseteq \bigsqcap C_i$.

A set of $\mathcal{EL}$ concepts C_i for $1 \leq i \leq n$ is $\mathtt{MGS}(\mathcal{T}, \Sigma, A)$ if the following conditions are fulfilled:

- *$sig(C_i) \subseteq \Sigma$ for all i,*
- *$\mathcal{T} \models A \sqsubseteq \bigsqcap C_i$ and $\mathcal{T} \not\models \bigsqcap C_i \sqsubseteq A$*
- *for all $\mathcal{EL}$ concepts D with $sig(D) \subseteq \Sigma$ holds:*
  $\mathcal{T} \models A \sqsubseteq D$, iff $\mathcal{T} \models \bigsqcap C_i \sqsubseteq D$.

We denote $\mathtt{MSS}(A)$ and $\mathtt{MGS}(A)$ expressed in logic L by $\mathtt{MSS}^L(A)$ and $\mathtt{MGS}^L(A)$. If $\mathtt{MGS}(\mathcal{T}, \Sigma, A)$ consists of several incomparable disjuncts C_i, it cannot be expressed by an $\mathcal{EL}$ concept. However, in the following, it will come into notice that this is not further problematic for the computation of UI, since the disjunction appears only on the left-hand side and can therefore be expressed by the means of several inclusion axioms. If the TBox $\mathcal{T}$ and the signature Σ do not change, we omit them and simply write $\mathtt{MSS}(A)$ and $\mathtt{MGS}(A)$. For the remainder of this paper, we fix $\mathcal{T}$ to be a general $\mathcal{EL}$ TBox and Σ a signature.

4 Normalization

In order to simplify the computation of MGS and MSS, we apply the following normalization thereby restricting the syntactic form of $\mathcal{T}$. Analogously to the normalization employed in other approaches ([1], [6], [7]), we decompose complex axioms into syntactically simple ones. The decomposition is realized recursively by replacing expressions $B_1 \sqcap ... \sqcap B_n$ and $\exists r.B$ with fresh concept symbols until each axiom in $\mathcal{T}$ is one of $\{A \sqsubseteq B, A \equiv B_1 \sqcap ... \sqcap B_n, A \equiv \exists r.B\}$, where $A, B, B_i \in N_C \cup \{\top\}$ and $r \in N_R$. For this purpose, we introduce a minimal required set of fresh concept symbols $A' \in N_D$ and the corresponding definition axioms $(A' \equiv C)$ for each of them. In what follows, we assume that knowledge bases are normalized and refer to $N_C \cup N_D$ as N_C. Since concept symbols in N_D are fresh, they do not appear in Σ and are therefore elements of the forgotten signature $\overline{\Sigma}$. Further, we assume that equivalent concept symbols have

Algorithm 1 computing $\texttt{MGS}_{\texttt{core}}(A)$ for an $\mathcal{EL}$ TBox $\mathcal{T}$ and a signature Σ

$M \leftarrow \bigcup \texttt{MGS}_{\texttt{cond}}(D, A), D \in N_C$ such that $\mathcal{T} \models D \sqsubseteq A, A \neq D$
for all $A \equiv \bigsqcap_{1 \leq i \leq n} B_i \in \mathcal{T}$ **do**
  $M \leftarrow M \cup \{\bigsqcap_{C \in \texttt{REDUCE}_{\texttt{MSS}}(\{C_i | 1 \leq i \leq n\})} C | (C_1, ..., C_n) \in \texttt{MGS}_{\texttt{cond}}(B_1, A) \times ... \times \texttt{MGS}_{\texttt{cond}}(B_n, A)\}$
end for
for all $A \equiv \exists r.B \in \mathcal{T}$ with $r \in \Sigma$ **do**
  $M \leftarrow M \cup \{\exists r.C | C \in \texttt{MGS}_{\texttt{cond}}(B, A)\}$
end for
return $\texttt{REDUCE}_{\texttt{MGS}}(M)$

Algorithm 2 computing $\texttt{MSS}_{\texttt{core}}(A)$ for an $\mathcal{EL}$ TBox $\mathcal{T}$ and a signature Σ

$M \leftarrow \bigcup \texttt{MSS}_{\texttt{cond}}(D, A), D \in N_C^+$ such that $\mathcal{T} \models A \sqsubseteq D, A \neq D$
for all $A \equiv \bigsqcap_{1 \leq i \leq n} B_i \in \mathcal{T}$ **do**
  $M \leftarrow M \cup \{C | C \in \texttt{MSS}_{\texttt{cond}}(B_i, A)\}$
end for
for all $A \equiv \exists r.B \in \mathcal{T}$ with $r \in \Sigma$ **do**
  $M \leftarrow M \cup \{\exists r. \bigsqcap_{C \in \texttt{MSS}_{\texttt{cond}}(B,A)} C\}$
end for
return $\texttt{REDUCE}_{\texttt{MSS}}(M)$

been replaced by a single representative of the corresponding equivalence class.[1] The following lemma summarizes the properties of normalized TBoxes.

Lemma 1. *Any $\mathcal{T}$ can be extended into a normalized TBox $\mathcal{T}'$ and each model of $\mathcal{T}$ can be extended into a model of $\mathcal{T}'$.*

Proof Sketch. All introduced concepts in N_D are defined in terms of concepts with $\text{sig}(C) \subseteq \text{sig}(\mathcal{T})$, therefore each model of $\mathcal{T}$ can be extended into a model of $\mathcal{T}'$. □

5 Computing MGS and MSS for Acyclic TBoxes

Given an acyclic, normalized $\mathcal{EL}$ TBox $\mathcal{T}$ and a signature Σ, Algorithms 1 and 2 compute for each $A \in N_C$ the elements of $\texttt{MGS}(A)$ and $\texttt{MSS}(A)$, respectively. The algorithms are derived from a Gentzen-style proof system and proceed along the definitions for A in $\mathcal{T}$ as well as the inferred inclusions between atomic concepts involving A. The computation is indirectly recursive and consists of the procedure $\texttt{MGS}_{\texttt{core}}$ ($\texttt{MSS}_{\texttt{core}}$) containing the core computation and procedure $\texttt{MGS}_{\texttt{cond}}$ ($\texttt{MSS}_{\texttt{cond}}$) realizing the termination of the computation: if the first parameter of $\texttt{MGS}_{\texttt{cond}}$ ($\texttt{MSS}_{\texttt{cond}}$) – a concept B – is in Σ, it returns B itself, which is the basecase of the computation; otherwise, it calls in turn $\texttt{MGS}_{\texttt{core}}(B)$ ($\texttt{MSS}_{\texttt{core}}(B)$). Thereby, $\texttt{MGS}_{\texttt{cond}}$ ($\texttt{MSS}_{\texttt{cond}}$) ensures that MGS and MSS only contain Σ-concepts. To avoid confusion, we denote $\texttt{MGS}(A)$ and $\texttt{MSS}(A)$ obtained using this simple definition of $\texttt{MGS}_{\texttt{cond}}$ ($\texttt{MSS}_{\texttt{cond}}$) by $\texttt{MGS}_{\text{acyc}}(A)$ and $\texttt{MSS}_{\text{acyc}}(A)$.

[1] The elimination of equivalent symbols does not affect the correctness or completeness of the uniform interpolation, since the removed symbols can easily be included into the resulting TBox.

Definition 3. *Let $\mathcal{T}$ an acyclic $\mathcal{EL}$ TBox and $A \in N_C$.* $\mathtt{MGS}_{acyc}(A) = \mathtt{MGS}_{\mathtt{core}}(A)$ *and* $\mathtt{MSS}_{acyc}(A) = \mathtt{MSS}_{\mathtt{core}}(A)$.

While this separation of concerns between $\mathtt{MGS}_{\mathtt{core}}(A)$ ($\mathtt{MSS}_{\mathtt{core}}(A)$) and $\mathtt{MGS}_{\mathtt{cond}}(B, A)$ ($\mathtt{MSS}_{\mathtt{cond}}(B, A)$) appears not necessary in the simple case of acyclic TBoxes, in the next section we extend the computation to the general case of GCIs by adding further conditions to $\mathtt{MGS}_{\mathtt{cond}}(B, A)$ ($\mathtt{MSS}_{\mathtt{cond}}(B, A)$) without changing the core computation presented in Algorithms 1 and 2. In particular, the role of second parameter of $\mathtt{MGS}_{\mathtt{cond}}$ will become clear.

The functions $\mathtt{REDUCE}_{\mathtt{MGS}}$ and $\mathtt{REDUCE}_{\mathtt{MSS}}$ eliminate redundancy within the computed results, which is not just an optimization, but will also play an important role when deciding the existence of a uniform interpolant in $\mathcal{EL}$. The two functions are defined as follows:

Definition 4. *Let $M = \{C_i | 0 \leq i \leq n\}$ be a set of $\mathcal{EL}$ concepts and $\mathcal{T}_e = \{\}$.*

$$\mathtt{REDUCE}_{\mathtt{MSS}}(M) = \{C_i \in M | \text{ there is no } C_j \in M \text{ such that } \mathcal{T}_e \models C_j \sqsubseteq C_i\}$$

$$\mathtt{REDUCE}_{\mathtt{MGS}}(M) = \{C_i \in M | \text{ the is no } C_j \in M \text{ such that } \mathcal{T}_e \models C_i \sqsubseteq C_j\}$$

Both, $\mathtt{REDUCE}$ and $\mathtt{REDUCE}_C$, can be easily realized using standard reasoning procedures in $\mathcal{EL}_{\mu(\sqcup),\nu}$, which is known to be decidable in *ExpTime* [4]. It is easy to see that, in case of an acyclic TBox $\mathcal{T}$, both algorithms terminate, while, in case of cyclic terminologies, the algorithms do not need to terminate. In the next section, we extend the computation to be applicable to general TBoxes and ensure the termination also for this case.

6 MGS and MSS for General TBoxes

As already mentioned, the computation based on the simple version of $\mathtt{MGS}_{\mathtt{cond}}(B, A)$ and $\mathtt{MSS}_{\mathtt{cond}}(B, A)$ does not always terminate in case of general TBoxes such as the TBox in Example 1. In order to exactly specify the case, in which Algorithms 1 and 2 do not terminate, we use the following graph structures representing the possible flow of computation of $\mathtt{MGS}_{\mathtt{core}}$ and $\mathtt{MSS}_{\mathtt{core}}$ for a particular TBox $\mathcal{T}$ (independent from a particular signature):

Definition 5. *The* MSS- *and* MGS-*graphs $\mathcal{A}_{\mathtt{MSS}}(\mathcal{T})$ and $\mathcal{A}_{\mathtt{MGS}}(\mathcal{T})$ are defined as*

- $\mathcal{A}_{\mathtt{MSS}}(\mathcal{T}) = (\Gamma_{\mathtt{MSS}}, Q, E_{\mathtt{MSS}})$ *with the set of edge labels* $\Gamma_{\mathtt{MSS}} = N_R \cup \{\sqsubseteq\}$, *the set of states* $Q = N_C$ *and the set of edges* $E_{\mathtt{MSS}} = \{(A, r, B) | A \equiv \exists r.B \in \mathcal{T}\} \cup \{(A, \sqsubseteq, B) | \mathcal{T} \models A \sqsubseteq B, A \neq B\}$, *where* $A, B \in Q$ *and* $r \in \Gamma_{\mathtt{MSS}}$.
- $\mathcal{A}_{\mathtt{MGS}}(\mathcal{T}) = (\Gamma_{\mathtt{MGS}}, Q, E_{\mathtt{MGS}})$ *with the set of edge labels* $\Gamma_{\mathtt{MGS}} = N_R \cup \{\sqsupseteq, \sqcap\}$, *the set of states* $Q = N_C$ *and the set of edges* $E_{\mathtt{MGS}} = \{(A, r, B) | A \equiv \exists r.B \in \mathcal{T}\} \cup \{(A, \sqsupseteq, B) | \mathcal{T} \models A \sqsupseteq B, A \neq B\} \cup \{(A, \sqcap, B) | A \equiv B \sqcap C \in \mathcal{T}$ *for any conjunction* C *of elements from* $Q\}$, *where* $A, B \in Q$ *and* $r \in \Gamma_{\mathtt{MGS}}$.

The two graphs can be constructed in linear time after the classification of the normalized TBox is finished. For $X \in \{\mathtt{MGS}, \mathtt{MSS}\}$, we denote the set of the paths in $\mathcal{A}_X(\mathcal{T}, \Sigma)$ from A to B as $L_X(A, B)$ and the set of the intersection-free paths from node

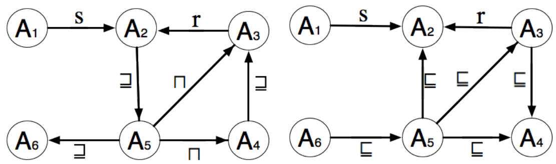

Fig. 1. MGS-graph (left) and MSS-graph (right) of $\mathcal{T}$.

A to itself as $L^1_X(A, A)$, i.e., such paths not contain any node except for A more than once. As illustrated by the example below, cyclic paths in $\mathcal{A}_{\mathtt{MSS}}(\mathcal{T})$ and $\mathcal{A}_{\mathtt{MGS}}(\mathcal{T})$ do not necessarily coincide.

Example 2. The corresponding MGS- and MSS-graphs of the TBox $\mathcal{T} = \{A_1 \equiv \exists s.A_2, A_3 \equiv \exists r.A_2, A_3 \sqsubseteq A_4, A_5 \equiv A_3 \sqcap A_4, A_5 \sqsubseteq A_2, A_5 \sqsubseteq A_6\}$ are shown in Fig. 1.

Given $\mathcal{A}_{\mathtt{MSS}}(\mathcal{T})$ and $\mathcal{A}_{\mathtt{MGS}}(\mathcal{T})$, we can easily determine for a particular signature Σ, whether the computation of the UI by the means of Algorithms 1 and 2 with the simple version of $\mathtt{MGS}_{\mathtt{cond}}(B, A)$ and $\mathtt{MSS}_{\mathtt{cond}}(B, A)$ terminates: if neither $\mathcal{A}_{\mathtt{MSS}}(\mathcal{T})$ nor $\mathcal{A}_{\mathtt{MGS}}(\mathcal{T})$ contain cyclic paths formed only by concepts from $\overline{\Sigma}$. Note that other cycles do not lead to a non-termination, since $\mathtt{MGS}_{\mathtt{cond}}(B, A) = \{B\}$ for any $B \in \Sigma$ and $A \in N_C$, i.e., the computation terminates. We denote such cyclic intersection-free paths from A containing only concepts from $\overline{\Sigma}$ by $L^{1,\overline{\Sigma}}_X(A, A)$ and the sets of concepts involved in such cycles by $\overline{\Sigma}_{C,\mathtt{MGS}} = \{A | A \in \overline{\Sigma}, L^{1,\overline{\Sigma}}_{\mathtt{MGS}}(A, A) \neq \emptyset\}$ and $\overline{\Sigma}_{C,\mathtt{MSS}} = \{A | A \in \overline{\Sigma}, L^{1,\overline{\Sigma}}_{\mathtt{MSS}}(A, A) \neq \emptyset\}$.

In order to be able to compute MSS and MGS also in case of $\overline{\Sigma}_{C,\mathtt{MSS}} \cup \overline{\Sigma}_{C,\mathtt{MGS}} \neq \emptyset$, we extend $\mathtt{MGS}_{\mathtt{cond}}(A, B)$ and $\mathtt{MSS}_{\mathtt{cond}}(A, B)$ by introducing a new condition for concepts $A \in \overline{\Sigma}_{C,\mathtt{MSS}} \cup \overline{\Sigma}_{C,\mathtt{MGS}}$. Here, we require the second parameter B to determine when the quantification of the fixpoint expressions is necessary. If $\mathtt{MGS}_{\mathtt{cond}}$ or $\mathtt{MSS}_{\mathtt{cond}}$ is called from outside of the corresponding cycles ($\overline{\Sigma}_{C,\mathtt{MGS}}$ for $\mathtt{MGS}_{\mathtt{cond}}$ and $\overline{\Sigma}_{C,\mathtt{MSS}}$ for $\mathtt{MSS}_{\mathtt{cond}}$), we return the complete fixpoint expression in its quantified form. Otherwise, we prefer to return the simplest possible value necessary, which can then be used as a part of a more complex, quantified concept expression. This second parameter is used by the caller – $\mathtt{MGS}_{\mathtt{core}}$ or $\mathtt{MSS}_{\mathtt{core}}$ – to pass its own parameter to the called $\mathtt{MGS}_{\mathtt{cond}}$ or $\mathtt{MSS}_{\mathtt{cond}}$ and let it then decide, whether to return a quantified fixpoint expression or a non-quantified one.

Definition 6. *Let n, m be the cardinality of $\overline{\Sigma}_{C,\mathtt{MSS}}$ and $\overline{\Sigma}_{C,\mathtt{MGS}}$, respectively. Further, let $A_i \in \overline{\Sigma}_{C,\mathtt{MSS}}$ with $0 \leq i \leq n$ and $A_j \in \overline{\Sigma}_{C,\mathtt{MGS}}$ with $0 \leq j \leq m$. Let $\{X(A_i) | A_i \in \overline{\Sigma}_{C,\mathtt{MSS}}\}$ and $\{Y(A_j) | A_j \in \overline{\Sigma}_{C,\mathtt{MGS}}\}$ be two disjoint sets of concept variables. Then, we define for each $A_i \in \overline{\Sigma}_{C,\mathtt{MSS}}$ and each $A_j \in \overline{\Sigma}_{C,\mathtt{MGS}}$:*

$$N(A_i) = \nu_i X(A_1), ..., X(A_n). \sqcap_{C \in \mathtt{MSS}_{\mathtt{core}}(A_1)} C, ..., \sqcap_{C \in \mathtt{MSS}_{\mathtt{core}}(A_n)} C$$

$$M(A_j) = \mu_j Y(A_1), ..., Y(A_m). \sqcup_{C \in \mathtt{MGS}_{\mathtt{core}}(A_1)} C, ..., \sqcup_{C \in \mathtt{MGS}_{\mathtt{core}}(A_m)} C.$$

$\mathtt{MSS}_{\mathtt{cond}}(A, B)$ and $\mathtt{MGS}_{\mathtt{cond}}(A, B)$ for any $A, B \in N_C$ is then given by:

$$\texttt{MSS}_{\texttt{cond}}(A,B) = \begin{cases} \{A\} & \textit{if } A \in \Sigma \\ \{X(A)\} & \textit{if } A \in \overline{\Sigma}_{C,\texttt{MSS}}, \\ & B \in \overline{\Sigma}_{C,\texttt{MSS}} \\ \{N(A)\} & \textit{if } A \in \overline{\Sigma}_{C,\texttt{MSS}}, \\ & B \notin \overline{\Sigma}_{C,\texttt{MSS}} \\ \texttt{MSS}_{\texttt{core}}(A) & \textit{otherwise} \end{cases} \qquad \texttt{MGS}_{\texttt{cond}}(A,B) = \begin{cases} \{A\} & \textit{if } A \in \Sigma \\ \{Y(A)\} & \textit{if } A \in \overline{\Sigma}_{C,\texttt{MGS}}, \\ & B \in \overline{\Sigma}_{C,\texttt{MGS}} \\ \{M(A)\} & \textit{if } A \in \overline{\Sigma}_{C,\texttt{MGS}}, \\ & B \notin \overline{\Sigma}_{C,\texttt{MGS}} \\ \texttt{MGS}_{\texttt{core}}(A) & \textit{otherwise} \end{cases},$$

and $\texttt{MGS}(A) = \texttt{MGS}_{\texttt{cond}}(A, \top)$ *and* $\texttt{MSS}(A) = \texttt{MSS}_{\texttt{cond}}(A, \top)$.

Note that, in case of an acyclic TBox, $\texttt{MGS}(A)$ coincides with $\texttt{MGS}_{\text{acyc}}(A)$ described in Section 5, and the same holds for $\texttt{MSS}(A)$.

Theorem 1 (Termination). *Let $A \in N_C$. The computation of $\texttt{MSS}(A)$ and $\texttt{MGS}(A)$ always terminates in at most exponential time.*

Proof Sketch. We first show by induction in case $\overline{\Sigma}_{C,\texttt{MSS}} = \emptyset$ that the computation of $\texttt{MSS}(A)$ for each $A \in N_C$ terminates. Then, we consider the case $\overline{\Sigma}_{C,\texttt{MSS}} \neq \emptyset$. $\texttt{MSS}_{\texttt{cond}}$ encapsulates all concepts in $\overline{\Sigma}_{C,\texttt{MSS}}$ into a single computational unit with direct or indirect incoming dependencies from concepts referencing concepts in $\overline{\Sigma}_{C,\texttt{MSS}}$ and direct or indirect outgoing dependencies to concepts referenced from any concept in $\overline{\Sigma}_{C,\texttt{MSS}}$. These two sets of referencing and referenced concepts are disjoint by definition of cyclicity. In the computation of $N(A)$, either concept variables or results of acyclic computations of $\texttt{MSS}(B)$ for B not referencing $\overline{\Sigma}_{C,\texttt{MSS}}$ are used, therefore each computation terminates. Once $N(A)$ is computed, references to $A \in \overline{\Sigma}_{C,\texttt{MSS}}$ do not require further computation and the remaining computation terminates as shown in case $\overline{\Sigma}_{C,\texttt{MSS}} \neq \emptyset$. Since the structure of $\texttt{MGS}_{\texttt{cond}}$ and $\texttt{MSS}_{\texttt{cond}}$ is analogous and $\texttt{MGS}_{\texttt{core}}$ also only iterates through the finite input directly, the argumentation for the termination of $\texttt{MGS}$ is identical. The complexity is due to the conjunction constructs introduced in line 3 of Algorithm 1. □

Theorem 2 (Correctness $\texttt{MSS}$ and $\texttt{MGS}$). *Let $A \in N_C$. The computed $\texttt{MSS}(A)$ and $\texttt{MGS}(A)$ satisfy the conditions stated in Definition 2.*

The proof of Theorem 2 is based on a Gentzen-style proof system for normalized TBoxes.

7 Computing Uniform Interpolants

Given $\texttt{MGS}(A)$ and $\texttt{MSS}(A)$ for each $A \in N_C$, we can compute the UI in $\mathcal{EL}_{\mu(\sqcup),\nu}$ as follows:

Definition 7. $\texttt{UI}(\mathcal{T},\Sigma) = \texttt{UI}_{\Sigma,\texttt{MSS}}(\mathcal{T},\Sigma) \cup \texttt{UI}_{\Sigma,\texttt{MGS}}(\mathcal{T},\Sigma) \cup \texttt{UI}_{\overline{\Sigma}}(\mathcal{T},\Sigma)$ *with*

- $\texttt{UI}_{\Sigma,\texttt{MSS}}(\mathcal{T},\Sigma) = \{A \sqsubseteq D | A \in N_C \cap \Sigma, D \in \texttt{MSS}(A)\}$,
- $\texttt{UI}_{\Sigma,\texttt{MGS}}(\mathcal{T},\Sigma) = \{C \sqsubseteq A | A \in N_C \cap \Sigma, C \in \texttt{MGS}(A)\}$,
- $\texttt{UI}_{\overline{\Sigma}}(\mathcal{T},\Sigma) = \{C \sqsubseteq D|$ *there is* $A \in N_C \cap \overline{\Sigma}$*, such that* $C \in \texttt{MGS}(A)$ *and* $D \in \texttt{MSS}(A)\}$.

Now, we have to prove that $\mathtt{UI}(\mathcal{T},\Sigma) \equiv^c_\Sigma \mathcal{T}$, i.e., the TBox $\mathtt{UI}(\mathcal{T},\Sigma)$ is in fact a uniform interpolant of $\mathcal{T}$ w.r.t. Σ.

Theorem 3 (UI). $\mathtt{UI}(\mathcal{T},\Sigma)$ *always exists and it holds that* $\mathtt{UI}(\mathcal{T},\Sigma) \equiv^c_\Sigma \mathcal{T}$.

The proof of Theorem 3 is also based on a Gentzen-style proof system for normalized TBoxes.

Deciding the Existence of UI in $\mathcal{EL}$

Clearly, if $\mathcal{T}$ does not contain pure $\overline{\Sigma}$ cycles, the $\mathtt{UI}(\mathcal{T},\Sigma)$ only contains $\mathcal{EL}$ constructs and, therefore, a $\mathtt{UI}$ in $\mathcal{EL}$ exists. This would be a sufficient, but not necessary criterion for the existence of a UI. From Definition 7, we can deduce a very general form of criterion requiring the deductive closure of any UI[2] to contain an (arbitrary) finite $\mathcal{EL}$ justification for the set of all non-$\mathcal{EL}$ axioms in the $\mathtt{UI}(\mathcal{T},\Sigma)$. Interestingly, given the $\mathcal{EL}$ TBox $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)$ obtained by extracting the $\mathcal{EL}$ part of each fixpoint concept within the non-mutual representation of $\mathtt{UI}(\mathcal{T},\Sigma)$, this criterion can be easily checked, since it is equivalent to a very simple criterion, which is an immediate consequence of the following theorem:

Theorem 4 (Existence). *Let* $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)$ *be the* $\mathcal{EL}$ *TBox obtained by extracting the* $\mathcal{EL}$ *part of each fixpoint concept within the non-mutual representation of* $\mathtt{UI}(\mathcal{T},\Sigma)$ *and let* $\mathcal{T}'$ *be an* $\mathcal{EL}$ *TBox with* $sig(\mathcal{T}') \subseteq \Sigma$ *such that* $\mathcal{T}' \equiv^c_\Sigma \mathcal{T}$. *Then* $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma) \equiv \mathcal{T}'$.

The theorem claims that, if a finite $\mathcal{EL}$ justification for the set of all non-$\mathcal{EL}$ axioms in $\mathtt{UI}(\mathcal{T},\Sigma)$ exists, it is already a contained in the non-mutual representation of $\mathtt{UI}(\mathcal{T},\Sigma)$. Subsequently, a UI of $\mathcal{T}$ w.r.t. Σ in $\mathcal{EL}$ exists, iff $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma) \models \mathtt{UI}(\mathcal{T},\Sigma)$. The proof of this theorem is based on the ideas stated in Lemmas 2 and 3, which show that there is a close relation between the existence of a UI in $\mathcal{EL}$ and redundancy in $\mathtt{UI}(\mathcal{T},\Sigma)$.

Lemma 2. *Let* $\mathcal{T}'$ *be an* $\mathcal{EL}$ *TBox with* $sig(\mathcal{T}') \subseteq \Sigma$ *such that* $\mathcal{T}' \equiv^c_\Sigma \mathcal{T}$. *Further, let* $A \in \overline{\Sigma}_{C,\mathtt{MGS}}$ *with* $C_1 \in \mathtt{MGS}(A)$ *and* $C_2 \in \mathtt{MSS}(A)$. *Then there is an* $\mathcal{EL}$ *concept* C' *such that*

- $\mathcal{T} \not\models C' \equiv C_1$ *and* $\mathcal{T} \not\models C' \equiv C_2$
- $\{\} \models C_1 \sqsubseteq C'$
- $\mathtt{UI}(\mathcal{T},\Sigma) \models C' \sqsubseteq C_2$.

Let $A \in \overline{\Sigma}_{C,\mathtt{MSS}}$ *with* $C_1 \in \mathtt{MGS}(A)$ *and* $C_2 \in \mathtt{MSS}(A)$. *Then there is an* $\mathcal{EL}$ *concept* C' *such that*

- $\mathcal{T} \not\models C' \equiv C_1$ *and* $\mathcal{T} \not\models C' \equiv C_2$
- $\{\} \models C' \sqsubseteq C_2$
- $\mathtt{UI}(\mathcal{T},\Sigma) \models C_1 \sqsubseteq C'$.

[2] The deductive closure is the same for any $\mathtt{UI}$ by definition.

Proof Sketch. Consider $C_1 = \mu X.(A \sqcup \exists r.X)$, which is the simplest possible non-$\mathcal{EL}$ concept in $\mathtt{MGS}$. C_1 is semantically equivalent to an infinite disjunction of more and more specific $\mathcal{EL}$ concepts. The language constructs of $\mathcal{EL}$ do not allow us to specify a concept, which captures exactly the subset of the interpretation domain $C_1^{\mathcal{I}}$ in all models. Let C_2 be an arbitrary concept with $\mathtt{UI}(\mathcal{T},\Sigma) \models C_1 \sqsubseteq C_2$. If $C_1 \sqsubseteq C_2$ is a consequence of $\mathcal{T}'$, then there must be an $\mathcal{EL}$ concept C_1', which subsumes $C_1^{\mathcal{I}}$ in all models. Since $\mathcal{T}'$ is a finite $\mathcal{EL}$ TBox, it must hold that $\mathcal{T}' \models C_1 \sqsubseteq C_1'$, i.e., the latter inclusion axiom must be derived from the finite $\mathcal{EL}$ TBox itself (e.g., $C_1' = B$ with $\{\exists r.B \sqsubseteq B, A \sqsubseteq B\} \in \mathcal{T}'$). Moreover $C_1' \sqsubseteq C_2$ must have a justification in $\mathcal{T}'$ consisting of finitely many $\mathcal{EL}$ axioms. The same argumentation applies to C_2 as a concept with greatest fixpoint constructs. □

The above proof is the first step towards a connection between the redundancy in $\mathtt{UI}(\mathcal{T},\Sigma)$ and the existence of a UI in $\mathcal{EL}$. Since $\{C_1 \sqsubseteq C', C' \sqsubseteq C_2\} \models C_1 \sqsubseteq C_2$ and any minimal justification of $\{C_1 \sqsubseteq C', C' \sqsubseteq C_2\}$ in any $\mathcal{T}'$ does not contain $C_1 \sqsubseteq C_2$, it also holds that $\mathtt{UI}(\mathcal{T},\Sigma) \cup \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma) \setminus \{C_1 \sqsubseteq C_2\} \models C_1 \sqsubseteq C_2$. Therefore, if $\mathcal{T}'$ exists, each non-$\mathcal{EL}$ axiom is redundant, i.e., it could be removed from $\mathtt{UI}(\mathcal{T},\Sigma) \cup \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)$ without losing any consequences. To avoid confusion, we denote the non-mutual representation of $\mathtt{MSS}(A)$ and $\mathtt{MGS}(A)$ with the corresponding $\mathcal{EL}$ parts explicitly appearing outside of all fixpoint quantifiers by $\mathtt{MSS}(A) \cup \mathtt{EL}(\mathtt{MSS}(A))$ and $\mathtt{MGS}(A) \cup \mathtt{EL}(\mathtt{MGS}(A))$. The functions $\mathtt{REDUCE}_{\mathtt{MSS}}$ and $\mathtt{REDUCE}_{\mathtt{MGS}}$ have to be applied also when computing $\mathtt{MSS}(A) \cup \mathtt{EL}(\mathtt{MSS}(A))$ and $\mathtt{MGS}(A) \cup \mathtt{EL}(\mathtt{MGS}(A))$. Therefore, the redundancy can only appear during the construction of $\mathtt{UI}(\mathcal{T},\Sigma) \cup \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)$. From the definition of $\mathtt{MGS}$ and $\mathtt{MSS}$ follows that the sets $\mathtt{UI}_{\Sigma,\mathtt{MSS}}(\mathcal{T},\Sigma)$ and $\mathtt{UI}_{\Sigma,\mathtt{MGS}}(\mathcal{T},\Sigma)$ in Definition 7 cannot be redundant if the sets $\mathtt{MSS}(A) \cup \mathtt{EL}(\mathtt{MSS}(A))$ and $\mathtt{MGS}(A) \cup \mathtt{EL}(\mathtt{MGS}(A))$ contain only incomparable elements. Therefore, it remains to consider the redundancy introduced during the construction of $\mathtt{UI}_{\overline{\Sigma}}(\mathcal{T},\Sigma)$. We denote by $P_{\overline{\Sigma}} = \{(C_1, C_2) |$ there is $A \in \overline{\Sigma}$ s.t. $C_1 \in \mathtt{MGS}(A) \cup \mathtt{EL}(\mathtt{MGS}(A)), C_2 \in \mathtt{MSS}(A) \cup \mathtt{EL}(\mathtt{MSS}(A))\}$ the set of all concept pairs relevant for the construction of $\mathtt{UI}_{\overline{\Sigma}}(\mathcal{T},\Sigma)$ and the subset of $P_{\overline{\Sigma}}$ containing the "redundant" concept pairs by $\mathcal{R} = \{(C_1, C_2) \in P_{\overline{\Sigma}} | (\mathtt{UI}(\mathcal{T},\Sigma) \cup \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)) \setminus \{C_1 \sqsubseteq C_2\} \models C_1 \sqsubseteq C_2\}$. I.e., $\mathcal{R}$ is the set of concept pairs that are potentially nonessential for the construction of a $\mathtt{UI}$ due to entailment of the corresponding inclusion axiom by the remainder of a $\mathtt{UI}$ if the axiom itself is omitted. Due to possible dependencies between the elements of $\mathcal{R}$, there may be several different maximal subsets M of $\mathcal{R}$ such that $(\mathtt{UI}(\mathcal{T},\Sigma) \cup \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)) \setminus \{C_1 \sqsubseteq C_2 | (C_1, C_2) \in M\} \models \mathtt{UI}(\mathcal{T},\Sigma)$. We denote the set of all such maximal subsets of $\mathcal{R}$ as $\mathcal{R}_{\mathtt{MAX}} = \{M | M \subseteq \mathcal{R}, (\mathtt{UI}(\mathcal{T},\Sigma) \cup \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)) \setminus \{C_1 \sqsubseteq C_2 | (C_1, C_2) \in M\} \models \mathtt{UI}(\mathcal{T},\Sigma)$, for all $(C_1', C_2') \in P_{\overline{\Sigma}} \setminus M$ holds $(\mathtt{UI}(\mathcal{T},\Sigma) \cup \mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma)) \setminus (\{C_1' \sqsubseteq C_2'\} \cup \{C_1 \sqsubseteq C_2 | (C_1, C_2) \in M\}) \not\models \mathtt{UI}(\mathcal{T},\Sigma)\}$. The next lemma states that if a concept pair with at least one non-$\mathcal{EL}$ concept is contained in one set $M \in \mathcal{R}_{\mathtt{MAX}}$, it is contained in all $M \in \mathcal{R}_{\mathtt{MAX}}$.

Lemma 3. *Let $\mathcal{T}'$ be an $\mathcal{EL}$ TBox with $sig(\mathcal{T}') \subseteq \Sigma$ such that $\mathcal{T}' \equiv_{\Sigma}^{c} \mathcal{T}$. Further, let $A \in \overline{\Sigma}_{C,\mathtt{MSS}} \cup \overline{\Sigma}_{C,\mathtt{MGS}}$ with $C_1 \in \mathtt{MGS}(A)$ and $C_2 \in \mathtt{MSS}(A)$. Let let $M' \in \mathcal{R}_{\mathtt{MAX}}$ such that $(C_1, C_2) \in M'$. Then for each $M \in \mathcal{R}_{\mathtt{MAX}}$ holds $(C_1, C_2) \in M$.*

Note that all concept pairs with at least one non-$\mathcal{EL}$ concept are contained in the intersection of $\mathcal{R}_{\mathtt{MAX}}$, iff $\mathtt{UI}^{\mathcal{EL}}(\mathcal{T},\Sigma) \equiv \mathcal{T}'$. As a consequence of the above two lemmas and the fact that for any $(C_1, C_2) \in \mathcal{R}$ there exists at least one $M \in \mathcal{R}_{\mathtt{MAX}}$, it is sufficient to check whether all concept pairs with at least one non-$\mathcal{EL}$ concept are contained in $\mathcal{R}$ to determine whether the $\mathcal{T}'$ in Theorem 4 exists.

8 Summary

In this paper, we provide an *ExpTime* algorithm for computing a uniform interpolant of general $\mathcal{EL}$ terminologies preserving all $\mathcal{EL}$ concept inclusions for a particular signature based on the notion of *most general subconcepts* and *most specific superconcepts*. The result of the computation is expressed in logic $\mathcal{EL}_{\mu(\sqcup),\nu}$—an extension of $\mathcal{EL}$ with least fixpoint and greatest fixpoint constructors μ, ν as well as the disjunction used only on the left-hand side of concept inclusions. We also state the exact existence criteria for an $\mathcal{EL}$ interpolant and show how it can be obtained from the corresponding interpolant expressed in $\mathcal{EL}_{\mu(\sqcup),\nu}$.

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