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Binary Logics, Orthologics and their Relations to Normal Modal Logics

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ABSTRACT. We study the relation between the class Ext(**O**) of orthologics and the class Next(**KTB**) of normal modal logics over **KTB** by means of embeddings of logics. First we introduce *binary logics*, which are generalizations of orthologics, as logics that are embeddable into some normal modal logics. We investigate the embeddability relation between the class of binary logics and the class of all normal modal logics, and establish some preservation results. Then we analyze the relation between Ext(**O**) and Next(**KTB**) and show that there exists a continuum of normal modal logics over **KTB**, and that any tabular orthologic is embeddable into infinitely many normal modal logics over **KTB**.

1 Binary logics as generalizations of orthologics

In 1974, R.I.Goldblatt showed that the orthologic **O** can be embedded into the normal modal logic **KTB** by a translation of formulas [2]. Our aim is to investigate the structure of the lattice of orthologics and that of normal modal logics by means of embeddability relations between these classes of logics.

The "basic" orthologic **O** can be thought of as the logic of *ortholattices*. The term "ortholattice" is an abbreviation of *ortho-complemented lattice* and a prime example of an ortholattice is the lattice of all subspaces of a finite dimensional vector space with orthogonal complement.

The most characteristic feature of ortholattices is that they do not in general have the distributive law. Because of this feature, we face some serious difficulties in studying ortholattices and orthologics. One of such difficulties in syntactical analysis of orthologics is the lack of a good implication connective [6].

Although the success of Kripke semantics brought about remarkable developments in the research of modal logics, we do not know as much about normal modal logics containing the axiom **B** ($q \supset \Box\Diamond q$) as about intermediate logics and modal logics containing **S4**. This is mainly because *symmetric* Kripke frames are less tractable than *transitive* ones. A completely new idea

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seems to be needed to investigate the modal logics containing **B**.

In [2], Goldblatt formulated orthologics as sets of pairs of formulas, in his words, as *binary logics*, introduced Kripke-type semantics of orthologics, and proved Kripke completeness of the smallest orthologic **O**. He showed moreover that **O** has the finite model property and hence it is decidable, and that **O** is embeddable into the normal modal logic **KTB** by a translation of formulas which we call here η_1.

In view of this embeddability, we may ask the following questions.

(1) What logic is embeddable into **K** by a translation similar to η_1? Further, can we establish an embeddability relation between the class of extensions of that logic and the class $\mathrm{NEXT}(\mathbf{K})$?

(2) Can we extend the embeddability of **O** into **KTB** to the embeddability relation between the class $\mathrm{EXT}(\mathbf{O})$ of extensions of **O** and the class $\mathrm{NEXT}(\mathbf{KTB})$ of normal extensions of **KTB**?

To answer question (1) above, we will define a notion of *binary logics* in the same fashion as orthologics in [2], where the orthocomplement is not only a kind of negation-like connective, but it behaves like an antitone modal operator. By introducing a new translation η_0, we can show that the smallest binary logic $\mathbf{L_0}$ is embeddable into **K**. Then we will discuss the relation between $\mathrm{EXT}(\mathbf{L_0})$ and $\mathrm{NEXT}(\mathbf{K})$ by Kripke-type semantics to some extent, and establish some preservation theorems by this embeddability relation.

To attack question (2), we use the translation η_1 to discuss the relation between $\mathrm{EXT}(\mathbf{O})$ and $\mathrm{NEXT}(\mathbf{KTB})$ in general. In particular, we can show that there exists a continuum of normal modal logics over **KTB**, and that any tabular orthologic is embeddable into at least countably infinitely many normal modal logics over **KTB** .

In the remaining part of this section, we will introduce the two classes of propositional logics that we are concerned with. First of all, we define the notion of *binary logics* in our sense.

The language $\mathcal{L}$ for binary logics consists of:(1) a set of propositional variables $\{p_1, p_2, \ldots\}$, (2) a propositional constant $\perp$, (3) conjunction $\wedge$, (4) a unary connective $\boxminus$, and (5) a pair of parentheses (,). The set Φ of formulas of $\mathcal{L}$ is defined as usual. Binary logics are defined as sets of pairs of formulas on Φ as below:

DEFINITION 1 (Binary logics). A *binary logic* **L** on the set Φ of formulas is a subset of the product $\Phi \times \Phi$ (we write $\alpha \vdash_{\mathbf{L}} \beta$, instead of $\langle \alpha, \beta \rangle \in \mathbf{L}$) which includes the following axiom schemes and is closed under the following inference rules

Axiom schemes:

(Ax1)
$$\alpha \vdash_{\mathbf{L}} \alpha$$

(Ax2)
$$\alpha \wedge \beta \vdash_{\mathbf{L}} \alpha$$

(Ax3)
$$\alpha \wedge \beta \vdash_{\mathbf{L}} \beta$$

(Ax4)
$$\bot \vdash_{\mathbf{L}} \alpha$$

(Ax5)
$$\alpha \vdash_{\mathbf{L}} \boxminus\bot$$

Inference Rules:

(R1)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta \quad \beta \vdash_{\mathbf{L}} \gamma}{\alpha \vdash_{\mathbf{L}} \gamma}$$

(R2)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta \quad \alpha \vdash_{\mathbf{L}} \gamma}{\alpha \vdash_{\mathbf{L}} \beta \wedge \gamma}$$

(R3)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta}{\boxminus\beta \vdash_{\mathbf{L}} \boxminus\alpha}$$

(R4)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta}{\alpha[\sigma/p] \vdash_{\mathbf{L}} \beta[\sigma/p]}$$

In (R4), $[\sigma/p]$ is a uniform substitution operator. We sometimes write $\alpha \vdash \beta$, instead of $\alpha \vdash_{\mathbf{L}} \beta$ if no confusion occurs. For a subset Γ of Φ, we write $\Gamma \vdash_{\mathbf{L}} \alpha$ to mean that there exist finitely many formulas $\beta_1, \beta_2, \ldots, \beta_n \in \Gamma$ such that $\langle \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n, \alpha \rangle \in \mathbf{L}$. The *smallest* binary logic is denoted by $\mathbf{L_0}$.

Note that our language $\mathcal{L}$ does not have the disjunction connective, and that our system for binary logics does not include any axiom scheme or any inference rule for it. If we introduce disjunction to our logic and interpret it classically (as *or*), we obtain essentially the same logic as the classical logic with a modal operator. If we introduce disjunction as an abbreviation of $\boxminus(\boxminus\alpha \wedge \boxminus\beta)$, we obtain the logics over the orthologic $\mathbf{O}$. Our aim however is to find a class of logics which are weaker than orthologics and which correspond to members of the class of all normal modal logics by a translation similar to Goldblatt's η_1. Thus, there can be no room for disjunction in our language.

For a binary logic $\mathbf{L}$, $\langle \alpha, \beta \rangle \in \mathbf{L}$ means, roughly speaking, that "β *is deducible from* α in $\mathbf{L}$", that is, we use pairs of formulas to compensate for the lack of implication symbol. $\mathbf{L}$ is said to be *consistent* if there exists a pair of formulas $\langle \alpha, \beta \rangle$ such that $\langle \alpha, \beta \rangle \notin \mathbf{L}$. For a set of pairs of formulas $\Sigma = \{\langle \alpha_\lambda, \beta_\lambda \rangle \in \Phi^2 \mid \lambda \in \Lambda\}$, the smallest binary logic which includes both $\mathbf{L_0}$, and Σ as axiom schemes, is denoted by $\mathbf{L_0} + \Sigma$. For a binary logic $\mathbf{L}$, the class of binary logics containing $\mathbf{L}$ is denoted by $\textsc{Ext}(\mathbf{L})$. In our terminology, the smallest orthologic $\mathbf{O}$ is a binary logic which is defined as: $\mathbf{O} := \mathbf{L_0} + \{\langle p, \boxminus\boxminus p \rangle, \langle \boxminus\boxminus p, p \rangle, \langle p \wedge \boxminus p, \bot \rangle\}$.

For normal modal logics, we will follow basic notions and nomenclature from [1]. We recall here only a few bits of notation we use in this paper.

The modal language $\mathcal{ML}$ consists of:(1) a set of propositional variables $\{q_1, q_2, \ldots\}$, (2) a propositional constant $\bot$, (3) conjunction $\wedge$, (4) negation $\neg$, (5) necessity $\Box$, and (6) a pair of parentheses (,). The set of formulas of this language is denoted by Ψ. The smallest normal modal logic is denoted by $\mathbf{K}$. For a set Γ of formulas the smallest normal modal logic that contains Γ is denoted by $\mathbf{K} \oplus \Gamma$. For a modal logic $\mathbf{M}$, the class of all normal modal logics above $\mathbf{M}$ is denoted by $\textsc{Next}(\mathbf{M})$.

2 Embedding of binary logics into normal modal logics

A map η from Φ to Ψ is called a *translation* of formulas. For a binary logic $\mathbf{L}$ and a normal modal logic $\mathbf{M}$, we say that $\mathbf{L}$ is *embeddable* into $\mathbf{M}$ by a translation η when for any $\alpha, \beta \in \Phi$, $\langle \alpha, \beta \rangle \in \mathbf{L}$ if and only if $\eta(\alpha) \supset \eta(\beta) \in \mathbf{M}$. In this case, $\mathbf{M}$ is called a *modal companion* of $\mathbf{L}$.

In this section, we consider the embeddability relation between $\textsc{Ext}(\mathbf{L_0})$ and $\textsc{Next}(\mathbf{K})$ by a translation of formulas $\eta_0 : \Phi \to \Psi$ defined inductively as follows: $\eta_0(\bot) := \bot$, $\eta_0(p_i) := q_i$, $\eta_0(\alpha \wedge \beta) := \eta_0(\alpha) \wedge \eta_0(\beta)$, and $\eta_0(\boxminus\alpha) := \Box\neg\eta_0(\alpha)$. We use algebraic semantics first, to establish a general embeddability result.

DEFINITION 2 (Algebra for binary logic). $\mathfrak{A} = \langle \mathsf{A}, \cap, J, 0, 1 \rangle$ is an *algebra for binary logic* (ABL for short) if and only if $\langle \mathsf{A}, \cap, 0, 1 \rangle$ is a bounded meet semilattice and J is a unary operation on A satisfying

1. $J(0) = 1$
2. for any $x, y \in \mathsf{A}$, $x \leq y$ implies $J(y) \leq J(x)$

In an ABL $\mathfrak{A}$, a *partial order* $\leq$ can be defined as: $x \leq y$ if and only if $x \cap y = x$. A *valuation* on $\mathfrak{A}$ is an assignment $v : \Phi \to \mathfrak{A}$ such that $v(\bot) = 0$, $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$, and $v(\boxminus\alpha) = J(v(\alpha))$. A pair of formulas $\langle \alpha, \beta \rangle$ is *valid* in an ABL $\mathfrak{A}$ ($\mathfrak{A} \models \langle \alpha, \beta \rangle$ in symbols) if and only if $v(\alpha) \leq v(\beta)$ holds for any valuation $v : \Phi \to \mathfrak{A}$. Then, the algebraic completeness for the smallest binary logic $\mathbf{L_0}$ can be shown in the usual way. That is, for any formulas α, β, $\langle \alpha, \beta \rangle \in \mathbf{L_0}$ if and only if $\mathfrak{A} \models \langle \alpha, \beta \rangle$ for any ABL $\mathfrak{A}$. In a similar fashion, any class of ABLs can determine a binary logic. More precisely, for any ABL $\mathfrak{A}$, the binary logic characterized by this $\mathfrak{A}$ is denoted by $\mathbf{L}(\mathfrak{A})$, that is, $\mathbf{L}(\mathfrak{A}) := \{\langle \alpha, \beta \rangle \,|\, \mathfrak{A} \models \langle \alpha, \beta \rangle\}$. For a class $\mathcal{C}$ of ABLs, the binary logic $\mathbf{L}(\mathcal{C})$, that is characterized by the class $\mathcal{C}$, is defined as: $\mathbf{L}(\mathcal{C}) := \bigcap_{\mathfrak{A} \in \mathcal{C}} \mathbf{L}(\mathfrak{A})$.

DEFINITION 3 (Modal algebra). $\mathfrak{B} = \langle \mathsf{B}, \cap, \cup, -, I, 0, 1\rangle$ is a *modal algebra* if and only if it satisfies the following:

(1) $\langle \mathsf{B}, \cap, \cup, -, 0, 1\rangle$ is a Boolean algebra.

(2) I is a unary operation on B satisfying: for all $x, y \in \mathsf{B}$,

   (a) $I(1) = 1$

   (b) $I(x \cap y) = I(x) \cap I(y)$

Let $\mathfrak{B}$ be a modal algebra. A *modal valuation* $u : \Psi \to \mathfrak{B}$ is a function which satisfies the conditions: $u(\bot) = 0$, $u(A \wedge B) = u(A) \cap u(B)$, $u(\Box A) = I(u(A))$. A modal formula A is *valid* in a modal algebra $\mathfrak{B}$ ($\mathfrak{B} \models A$ in symbol) if and only if $u(A) = 1$ for any modal valuation $u : \Psi \to \mathfrak{B}$. Then it is easy to see that the logic $\mathbf{K}$ is characterized by the class of all modal algebras. That is, for any modal formula A, $A \in \mathbf{K}$ if and only if $\mathfrak{B} \models A$ for every modal algebra $\mathfrak{B}$. Any class of modal algebras determines a normal modal logic. The modal logic that is determined by a modal algebra $\mathfrak{B}$ is denoted by $\mathbf{M}(\mathfrak{B})$. For a class $\mathcal{D}$ of modal algebras, we put $\mathbf{M}(\mathcal{D}) := \bigcap_{\mathfrak{B} \in \mathcal{D}} \mathbf{M}(\mathfrak{B})$.

Note that for any binary logic $\mathbf{L}$, there exists a class $\mathcal{C}$ of ABLs such that $\mathbf{L} = \mathbf{L}(\mathcal{C})$. This is due to the fact that the Lindenbaum algebra for $\mathbf{L}$ characterizes the logic $\mathbf{L}$. Of course, such a class of ABLs is not uniquely determined. The situation is the same for any normal modal logic.

For a modal algebra $\mathfrak{B}$, we define an algebra $\widehat{\mathfrak{A}_0}(\mathfrak{B}) := \langle \mathsf{B}, \cap, J_{\mathfrak{B}}, 0, 1\rangle$, where $J_{\mathfrak{B}}(x) := I(-x)$. It is easily seen that $\widehat{\mathfrak{A}_0}(\mathfrak{B})$ is an ABL for any $\mathfrak{B}$. Then the following holds.

LEMMA 4. *For $\alpha, \beta \in \Phi$, $\widehat{\mathfrak{A}_0}(\mathfrak{B}) \models \langle \alpha, \beta\rangle$ if and only if $\mathfrak{B} \models \eta_0(\alpha) \supset \eta_0(\beta)$.*

Proof. Suppose $\mathfrak{B} \not\models \eta_0(\alpha) \supset \eta_0(\beta)$. Then there exists a modal valuation $u : \Psi \to \mathsf{B}$ such that $u(\eta_0(\alpha)) \not\leq u(\eta_0(\beta))$. Define a valuation $v : \Phi \to \mathsf{B}$ as: $v(p_i) := u(q_i)$. Then we can show by induction that $v(\varphi) = u(\eta_0(\varphi))$ for any $\varphi \in \Phi$. Therefore we have $v(\alpha) \not\leq v(\beta)$ in $\widehat{\mathfrak{A}_0}(\mathfrak{B})$. Similarly suppose $\widehat{\mathfrak{A}_0}(\mathfrak{B}) \not\models \langle \alpha, \beta\rangle$. Then there exists a valuation $v : \Phi \to \mathsf{B}$ such that $v(\alpha) \not\leq v(\beta)$. Define a modal valuation $u : \Psi \to \mathsf{B}$ as $u(q_i) := v(p_i)$. Then by the fact that $v(\varphi) = u(\eta_0(\varphi))$ for any $\varphi \in \Phi$, we have $u(\eta_0(\alpha)) \not\leq u(\eta_0(\beta))$. ■

Note that in the proof above, the fact that $v(\varphi) = u(\eta_0(\varphi))$ for any $\varphi \in \Phi$ shows that the construction $\widehat{\mathfrak{A}_0}(\cdot)$ is closely related to the translation η_0. By this lemma, the following theorems are proved.

THEOREM 5. *Let* $\mathbf{L}$ *be a binary logic and* $\mathbf{M}$ *a modal logic. Then the following two conditions are equivalent.*

(1) $\mathbf{L}$ *is embeddable into* $\mathbf{M}$ *by* η_0.

(2) There exists a class $\mathcal{D}$ *of modal algebras such that:*
(a) $\mathbf{M} = \mathbf{M}(\mathcal{D})$ *and (b)* $\mathbf{L} = \mathbf{L}(\{\widehat{\mathfrak{A}_0}(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{D}\})$.

Proof. From (2) to (1), it is immediate. For the converse direction, we take as $\mathcal{D}$ the class of modal algebras that determines the logic $\mathbf{M}$. This means that $\mathbf{M} = \mathbf{M}(\mathcal{D})$. Then we have to show (b). Take any $\langle\alpha, \beta\rangle \in \mathbf{L}$ and any $\mathfrak{B} \in \mathcal{D}$. Since $\mathbf{L}$ is embeddable into $\mathbf{M}$ by η_0, we have $\eta_0(\alpha) \supset \eta_0(\beta) \in \mathbf{M}$; that means $\mathfrak{B} \models \eta_0(\alpha) \supset \eta_0(\beta)$, and by Lemma 4, this is equivalent to $\widehat{\mathfrak{A}_0}(\mathfrak{B}) \models \langle\alpha, \beta\rangle$. For $\langle\alpha, \beta\rangle \notin \mathbf{L}$, by a similar argument, we can show that there exists a modal algebra $\mathfrak{B}$ such that $\widehat{\mathfrak{A}_0}(\mathfrak{B}) \not\models \langle\alpha, \beta\rangle$. ■

Given any ABL $\mathfrak{A}$, if we could construct a modal algebra $\mathfrak{B}$ such that $\widehat{\mathfrak{A}_0}(\mathfrak{B})$ is isomorphic to $\mathfrak{A}$, it would follow that every binary logic has a modal companion. Unfortunately, we have not been able to construct such a $\mathfrak{B}$ and thus it is still unknown whether modal companions always exist. In the other direction, however, every normal modal logic $\mathbf{M}$ has a corresponding binary logic which is embeddable into $\mathbf{M}$ by η_0, and this binary logic is determined uniquely by $\mathbf{M}$.

THEOREM 6. *For any modal logic* $\mathbf{M}$*, there exists a unique binary logic* $\mathbf{L}$ *such that* $\mathbf{L}$ *is embeddable into* $\mathbf{M}$ *by* η_0.

Proof. Put $\mathcal{D}_{\mathbf{M}} := \{\mathfrak{B} \mid \mathfrak{B} \models A \text{ for any } A \in \mathbf{M}\}$. Then it is easily seen by the above theorem that $\mathbf{L} = \mathbf{L}(\{\widehat{\mathfrak{A}_0}(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{D}_{\mathbf{M}}\})$ is embeddable into $\mathbf{M}$ by η_0. If a binary logic $\mathbf{L}'$ is also embeddable into $\mathbf{M}$ by η_0, then for any $\alpha, \beta \in \Phi$, $\langle\alpha, \beta\rangle \in \mathbf{L}'$ if and only if $\eta_0(\alpha) \supset \eta_0(\beta) \in \mathbf{M}$ if and only if $\langle\alpha, \beta\rangle \in \mathbf{L}$. This means that $\mathbf{L}' = \mathbf{L}$. ■

It is clear that if a binary logic $\mathbf{L}$ has a modal companion, then the set of all modal companions of $\mathbf{L}$ has the smallest element. But the largest element in this set does not always exist. As for a characterization of logics for which it does exist, we only know the following.

PROPOSITION 7. *Let* $\mathbf{L}$ *be a binary logic and* $MC_{\eta_0}(\mathbf{L})$ *the set of all modal companions of* $\mathbf{L}$. *Suppose* $MC_{\eta_0}(\mathbf{L}) \neq \emptyset$. *Then the greatest normal modal logic exists in* $MC_{\eta_0}(\mathbf{L})$ *if and only if* $\langle MC_{\eta_0}(\mathbf{L}), \subseteq\rangle$ *is directed.*

Proof. If the greatest element exists in $MC_{\eta_0}(\mathbf{L})$, then it is immediate that $\langle MC_{\eta_0}(\mathbf{L}), \subseteq\rangle$ is directed. Conversely, suppose $\langle MC_{\eta_0}(\mathbf{L}), \subseteq\rangle$ is directed. Put $\overline{\mathbf{M}} := \bigcup MC_{\eta_0}(\mathbf{L})$. Then it is trivial that $\mathbf{L}$ is embeddable into $\overline{\mathbf{M}}$ by η_0 and that $\overline{\mathbf{M}}$ is the greatest. And we can also show that $\overline{\mathbf{M}}$ is a normal modal logic because of directedness. ■

3 Kripke complete binary logics and their embedding into normal modal logics

3.1 Kripke-type semantics for binary logics

Apart from algebraic semantics, we can also employ a Kripke-type semantics for binary logics, and indeed we can prove that the binary logic $\mathbf{L_0}$ is complete with respect to it. Our semantics is a modification of orthoframes and orthomodels used by Goldblatt in [2]. Using this type of semantics we analyze the embeddability of *Kripke complete* binary logics into normal modal logics, and we prove the preservation of Kripke completeness and finite model property between them.

DEFINITION 8 (B-frame and B-model). A *B-frame* is a structure $\mathcal{G} = \langle W, S, Q\rangle$, where W is a non-empty set, S is a binary relation on W, and Q is a subset of $\mathcal{P}(W)$ that is closed under $\cap$ (intersection) and a unary operation J_S on $\mathcal{P}(W)$ defined as: $J_S(X) := \{a \in W \mid \forall b \in W, (b \in X \text{ implies } aSb)\}$ for $X \in \mathcal{P}(W)$. A *B-model* on a frame $\mathcal{G}$ is $\mathfrak{N} = \langle \mathcal{G}, U\rangle$, where U is a map that assigns to each propositional variable an element in Q. In a B-model $\mathfrak{N}$, U generates a relation $\models$ on $W \times \Phi$ as follows. Below, $\mathfrak{N} \models_a \alpha$ means that a formula α is true at a in a model $\mathfrak{N}$.

(1) $\mathfrak{N} \models_a p_i$ iff $a \in U(p_i)$.

(2) $\mathfrak{N} \not\models_a \bot$ for any $a \in W$.

(3) $\mathfrak{N} \models_a \alpha \wedge \beta$ iff $\mathfrak{N} \models_a \alpha$ and $\mathfrak{N} \models_a \beta$.

(4) $\mathfrak{N} \models_a \boxminus\alpha$ iff for any $b \in W$, $\mathfrak{N} \models_b \alpha$ implies aSb

In particular, any B-frame of the form $\langle W, S, \mathcal{P}(W)\rangle$ is called a *Kripke B-frame*, and denoted by $\langle W, S\rangle$. Let Γ be a set of formulas and α a formula. Γ *implies* α at $a \in W$ in $\mathfrak{N}$ ($\mathfrak{N} : \Gamma \models_a \alpha$) if and only if there exist $\beta_1, \beta_2, \ldots, \beta_n \in \Gamma$ such that $\mathfrak{N} \not\models_a \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n$, or $\mathfrak{N} \models_a \alpha$. Γ *implies* α in $\mathfrak{N}$ ($\mathfrak{N} : \Gamma \models \alpha$) if $\mathfrak{N} : \Gamma \models_a \alpha$ for all $a \in W$. Γ *implies* α in a B-frame $\mathcal{G}$ ($\mathcal{G} : \Gamma \models \alpha$) if and only if $\mathfrak{N} : \Gamma \models \alpha$ for any B-model $\mathfrak{N}$ on $\mathcal{G}$. Let $\mathcal{C}$ be a class of B-frames. Γ *implies* α in $\mathcal{C}$ ($\mathcal{C} : \Gamma \models \alpha$) if and only if $\mathcal{G} : \Gamma \models \alpha$ for any B-frame $\mathcal{G}$ in $\mathcal{C}$. We sometimes write $\mathfrak{N} \models_a \langle\alpha, \beta\rangle$ instead of $\mathfrak{N} : \{\alpha\} \models_a \beta$. $\mathcal{G} \models \langle \alpha, \beta \rangle$ and $\mathcal{C} \models \langle \alpha, \beta \rangle$ are used similarly.

We can prove the following completeness theorem of the binary logic $\mathbf{L_0}$ with respect to the class of all Kripke B-frames.

THEOREM 9 (Completeness for $\mathbf{L_0}$). *Let Θ be the set of all Kripke B-frames. For any set of formulas Γ and any formula α, $\Theta : \Gamma \models \alpha$ if and only if $\Gamma \vdash_{\mathbf{L_0}} \alpha$.*

The proof is omitted except for the construction of the canonical B-model. For a consistent binary logic $\mathbf{L}$, a non-empty set Γ of formulas is $\mathbf{L}$-*full* if Γ satisfies the following:

(a) there exists a formula α such that $\Gamma \nvdash_{\mathbf{L}} \alpha$,

(b) for $\alpha, \beta \in \Phi$, if $\alpha \in \Gamma$ and $\alpha \vdash_{\mathbf{L}} \beta$, then $\beta \in \Gamma$,

(c) for $\alpha, \beta \in \Phi$, if $\alpha, \beta \in \Gamma$, then $\alpha \wedge \beta \in \Gamma$.

Then the canonical B-model for a consistent binary logic is defined as follows.

DEFINITION 10 (Canonical model for binary logics). Let $\mathbf{L}$ be a consistent binary logic. $\mathfrak{N}_{\mathbf{L}} = \langle W_{\mathbf{L}}, S_{\mathbf{L}}, Q_{\mathbf{L}}, U_{\mathbf{L}} \rangle$ is the *canonical B-model for* $\mathbf{L}$ if it satisfies the following:

(1) $W_{\mathbf{L}} := \{a \subseteq \Phi(V) \mid a \text{ is } \mathbf{L}\text{-full}\}$.

(2) $S_{\mathbf{L}}$ is a binary relation on $W_{\mathbf{L}}$ defined as: $x S_{\mathbf{L}} y$ if and only if there exists $\varphi \in \Phi$, such that $\boxminus\varphi \in x$ and $\varphi \in y$.

(3) $Q_{\mathbf{L}} := \{\{x \in W_{\mathbf{L}} \mid \alpha \in x\} \mid \alpha \in \Phi\}$.

(4) $U_{\mathbf{L}}(p_i) := \{x \in W_{\mathbf{L}} \mid p_i \in x\}$

For any consistent binary logic, we can prove the completeness theorem with respect to a suitable class of B-frames by the standard technique of general frames for normal modal logics. That is, the following holds.

THEOREM 11. *Let $\mathbf{L} := \mathbf{L}_0 + \Sigma$ be a consistent binary logic. Then there exists a class $\mathcal{C}_{\mathbf{L}}$ of B-frames for $\mathbf{L}$ such that for $\Gamma \subseteq \Phi$ and $\alpha \in \Phi$, $\mathcal{C}_{\mathbf{L}} : \Gamma \models \alpha$ if and only if $\Gamma \vdash_{\mathbf{L}} \alpha$.*

To choose suitable B-frames for $\mathbf{L}$, we have to adjust Q in a B-frame depending on Σ.

We can also introduce the notion of Kripke completeness for binary logics as follows.

DEFINITION 12 (Kripke completeness for binary logics). A binary logic $\mathbf{L}$ is called *Kripke complete* if and only if the following are equivalent. For any pair of formulas $\langle\alpha,\beta\rangle$,

(1) $\langle\alpha,\beta\rangle \in \mathbf{L}$.

(2) For any Kripke B-frame $\mathcal{G}$, if $\mathcal{G}:\varphi \models \psi$ for any $\langle\varphi,\psi\rangle \in \mathbf{L}$, then $\mathcal{G}:\alpha \models \beta$

In other words, $\mathbf{L}$ is Kripke complete if and only if there exists a class $\mathcal{C}_{\mathbf{L}}$ of Kripke B-frames for $\mathbf{L}$ such that for any pair $\langle\alpha,\beta\rangle$ of formulas, $\langle\alpha,\beta\rangle \in \mathbf{L}$ is equivalent to $\mathcal{C}_{\mathbf{L}} \models \langle\alpha,\beta\rangle$.

Now we discuss the embeddability via Kripke-type semantics. Frames and models for normal modal logics are defined as usual. We also use standard notation, for example, $\mathcal{F} = \langle W,R,P\rangle$ for a (general) frame, $\mathfrak{M} = \langle\mathcal{F},V\rangle$ for a model, and $\mathcal{F} = \langle W,R,\mathcal{P}(W)\rangle = \langle W,R\rangle$ for a Kripke frame. Further, we denote $\mathfrak{M} \models_a A$ to mean that a formula A is true at a in a model $\mathfrak{M}$. A formula A is *valid* in a frame $\mathcal{F}$ ($\mathcal{F} \models A$) if and only if for any model $\mathfrak{M}$ on $\mathcal{F}$, $\mathfrak{M} \models_a A$ for any $a \in W$. A normal modal logic $\mathbf{M}$ is called *Kripke complete* when for each formula A, $A \in \mathbf{M}$ if and only if $\mathcal{F} \models A$ in any Kripke frame $\mathcal{F}$ such that $\mathcal{F} \models B$ for any $B \in \mathbf{M}$.

To understand how η_0 works, it is helpful to use the following pair of *transformations* of Kripke frames and Kripke B-frames. Let $\mathcal{K}$ be the class of all Kripke frames. For a binary relation R on a set W, $\check{R}$ denotes its complement as a subset $W \times W$. Transformations $\xi_0 : \Theta \to \mathcal{K}$ and $\xi_0^{-1} : \mathcal{K} \to \Theta$ are defined as follows: for Kripke B-model $\mathfrak{N} = \langle W,S,U\rangle$, $\xi_0(\mathfrak{N}) := \langle W,\check{S},V\rangle$, where $V(q_i) := U(p_i)$. Conversely, for Kripke model $\mathfrak{M} = \langle W,R,V\rangle$, $\xi_0^{-1}(\mathfrak{M}) := \langle W,\check{R},U\rangle$, where $U(p_i) := V(q_i)$. Clearly, ξ_0 and ξ_0^{-1} are mutually inverse, namely, $\xi_0^{-1} \circ \xi_0(\mathfrak{N}) = \mathfrak{N}$ for any $\mathfrak{N}$ and $\xi_0 \circ \xi_0^{-1}(\mathfrak{M}) = \mathfrak{M}$ for any $\mathfrak{M}$. Then η_0, ξ_0 and ξ_0^{-1} together have the following nice property.

LEMMA 13. *Let $\mathfrak{N} = \langle W,S,U\rangle$ be a Kripke B-model, and $\mathfrak{M} = \langle W,R,V\rangle$ a Kripke model.*

(1) For $\varphi \in \Phi$ and for any $x \in W$, $\mathfrak{N} \models_x \varphi$ if and only if $\xi_0(\mathfrak{N}) \models_x \eta_0(\varphi)$.

(2) For $\varphi \in \Phi$ and for any $x \in W$, $\mathfrak{M} \models_x \eta_0(\varphi)$ if and only if $\xi_0^{-1}(\mathfrak{M}) \models_x \varphi$.

Proof. Both can be proved by induction on the formula φ. We show only the case $\varphi \equiv \boxminus\psi$ for (1). Similar argument goes through for (2).
(1): By definition, $\mathfrak{N} \models_x \boxminus\psi$ if and only if for any $y \in W$, $\mathfrak{N} \models_y \psi$

implies xSy. Equivalently, for any $y \in W$, $x\check{S}y$ implies $\mathfrak{N} \not\models_y \psi$. By induction hypothesis, this is equivalent to the fact that for any $y \in W$, $x\check{S}y$ implies $\xi_0(\mathfrak{M}) \not\models_y \eta_0(\psi)$. In a normal modal logic, this means that $\xi_0(\mathfrak{M}) \models_x \Box\neg\eta_0(\psi)$, that is, $\xi_0(\mathfrak{M}) \models_x \eta_0(\boxminus\psi)$. ■

Preservation results for Kripke complete logics follow from the above lemma. For a Kripke B-frame $\mathcal{G} = \langle W, S\rangle$, we put $\xi_0(\mathcal{G}) := \langle W, \check{S}\rangle$ and for a class $\mathcal{C}$ of Kripke B-frames, and $\xi_0(\mathcal{C}) := \{\xi_0(\mathcal{G}) \mid \mathcal{G} \in \mathcal{C}\}$. For a Kripke complete binary logic $\mathbf{L}$, there exists a class $\mathcal{C}_{\mathbf{L}}$ of Kripke B-frames such that $\mathbf{L}$ is complete with respect to this $\mathcal{C}_{\mathbf{L}}$. For this $\mathbf{L}$, define a modal logic $\delta_{\xi_0}(\mathbf{L}) := \{A \in \Psi \mid \xi_0(\mathcal{C}_{\mathbf{L}}) \models A\}$. Then we have the following.

THEOREM 14. *Let* $\mathbf{L}$ *be a Kripke complete binary logic. Then,*

(1) $\mathbf{L}$ *is embeddable into* $\delta_{\xi_0}(\mathbf{L})$ *by* η_0.

(2) $\delta_{\xi_0}(\mathbf{L})$ *is Kripke complete with respect to the class* $\xi_0(\mathcal{C}_{\mathbf{L}})$.

(3) If $\mathbf{L}$ *has the finite model property, then so does* $\delta_{\xi_0}(\mathbf{L})$.

This theorem says that δ_{ξ_0} maps every Kripke complete binary logics to a Kripke complete modal logic as its modal companion, and it preserves finite model property.

On the other hand, for a Kripke frame $\mathcal{F} = \langle W, R\rangle$, we put $\xi_0^{-1}(\mathcal{F}) := \langle W, \check{R}\rangle$ and for a class $\mathcal{D}$ of Kripke frames, $\xi_0^{-1}(\mathcal{D}) := \{\xi_0^{-1}(\mathcal{F}) \mid \mathcal{F} \in \mathcal{D}\}$. For a Kripke complete normal modal logic $\mathbf{M}$, there exists a class $\mathcal{D}_{\mathbf{M}}$ of Kripke frames such that $\mathbf{M}$ is complete with respect to $\mathcal{D}_{\mathbf{M}}$. For this $\mathbf{M}$, define a binary logic $\varepsilon_{\xi_0^{-1}}(\mathbf{M}) := \{\langle\alpha, \beta\rangle \in \Phi^2 \mid \xi_0^{-1}(\mathcal{D}_{\mathbf{M}}) \models \langle\alpha, \beta\rangle\}$. Then similarly to Theorem 14, we have the following.

THEOREM 15. *Let* $\mathbf{M}$ *be a Kripke complete normal modal logic. Then,*

(1) $\varepsilon_{\xi_0^{-1}}(\mathbf{M})$ *is embeddable into* $\mathbf{M}$ *by* η_0.

(2) $\varepsilon_{\xi_0^{-1}}(\mathbf{M})$ *is Kripke complete with respect to the class* $\xi_0^{-1}(\mathcal{D}_{\mathbf{M}})$.

(3) If $\mathbf{M}$ *has the finite model property, then so does* $\varepsilon_{\xi_0^{-1}}(\mathbf{M})$.

Both theorems follow from Lemma 13. As we have seen above, Kripke complete logics possess very nice properties with respect to the embeddability relation. The reason is that transformations ξ_0 and ξ_0^{-1} are mutually inverse.

3.2 Correspondence theory for binary logics

Like *correspondence theory* for modal logic, we establish some correspondences between axiom schemes of binary logics and first order conditions on the relation S of B-frames. Let $\mathcal{G}$ be a B-frame and Form a first order formula in the language of frames. We write $\mathcal{G} \models \mathsf{Form}$ to mean that $\mathcal{G}$ satisfies the condition Form. First we give some examples of correspondence between axioms and first order conditions on Kripke B-frames.

PROPOSITION 16. *Let $\mathcal{G} = \langle W, S \rangle$ a Kripke B-frame.*

(1) $\mathcal{G} \models \langle p, \boxminus\boxminus p \rangle$ *if and only if* $\mathcal{G} \models \forall x, y(xSy \to ySx)$.

(2) $\mathcal{G} \models \langle p \wedge \boxminus p, \bot \rangle$ *if and only if* $\mathcal{G} \models \forall x \sim(xSx)$.

(3) $\mathcal{G} \models \langle p, \boxminus p \rangle$ *if and only if* $\mathcal{G} \models \forall x, y(xSy)$.

(4) $\mathcal{G} \models \langle \boxminus p, \boxminus\boxminus p \rangle$ *if and only if* $\mathcal{G} \models \forall x, y, z(xSy \,\&\, zSy \to xSz)$.

(5) $\mathcal{G} \models \langle p \wedge \boxminus p, \boxminus\boxminus p \rangle$ *if and only if* $\mathcal{G} \models \forall x(xSx \to \forall y(ySx \to xSy))$.

(6) $\mathcal{G} \models \langle \boxminus p \wedge \boxminus\boxminus p, \boxminus\boxminus\boxminus p \rangle$ *if and only if*
$\mathcal{G} \models \forall x[\forall y\{\forall z(xSz \to ySz) \to xSy\} \to$
$\forall u\{\forall v(\forall w(xSw \to vSx) \to uSv) \to xSu\}]$

Proof. We show only (5). Others can be proved similarly. Suppose that $\mathcal{G} \models \forall x(xSx \to \forall y(ySx \to xSy))$. Consider an arbitrary valuation U on $\mathcal{G}$ and a Kripke B-model $\mathfrak{N} = \langle \mathcal{G}, U \rangle$. Assume $\mathfrak{N} \models_a p \wedge \boxminus p$ for an element $a \in W$. Then $\mathfrak{N} \models_a p$ and $\mathfrak{N} \models_a \boxminus p$, and so, we have aSa. Take any $b \in W$, such that $\mathfrak{N} \models_b \boxminus p$. Then bSa holds, and by our supposition aSb also holds. Therefore we have $\mathfrak{N} \models_a \boxminus\boxminus p$. Thus we have $\mathcal{G} : p \models \boxminus\boxminus p$. Conversely, suppose $\mathcal{G} \not\models \forall x(xSx \to \forall y(ySx \to xSy))$. Then there exist $a, b \in W$ such that aSa, bSa, but $\sim(aSb)$. Define a valuation U such that $U(p) := \{a\}$. Then in the Kripke B-model $\mathfrak{N} = \langle \mathcal{G}, U \rangle$, $\mathfrak{N} \models_a p$ and $\mathfrak{N} \models_a \boxminus p$, and $\mathfrak{N} \models_b \boxminus p$.But since $\sim(aSb)$, $\mathfrak{N} \not\models_a \boxminus\boxminus p$. Thus we have $\mathcal{G} : p \wedge \boxminus p \not\models \boxminus\boxminus p$. ■

To say more for the case (5), let $\mathbf{L} = \mathbf{L_0} + \langle p \wedge \boxminus p, \boxminus\boxminus p \rangle$. Then it can be shown that the canonical frame for $\mathbf{L}$ satisfies the first order condition on the right hand side in (5). For, consider the canonical Kripke B-frame $\mathcal{G}_{\mathbf{L}} = \langle W_{\mathbf{L}}, S_{\mathbf{L}} \rangle$. Suppose that $xS_{\mathbf{L}}x$ for $x \in W_{\mathbf{L}}$, then there exists $\alpha \in \Phi$ such that $\boxminus\alpha \in x$ and $\alpha \in x$. Further assume that $yS_{\mathbf{L}}x$ for $y \in W_{\mathbf{L}}$, then there exists $\beta \in \Phi$ such that $\boxminus\beta \in y$ and $\beta \in x$. Here we have $\alpha \wedge \beta \in x$ and $\boxminus(\alpha \wedge \beta) \in x$, and so, $\boxminus\boxminus(\alpha \wedge \beta) \in x$ by the extra axiom for $\mathbf{L}$. On the other hand, $\boxminus(\alpha \wedge \beta) \in y$ also holds. Therefore $xS_{\mathbf{L}}y$. Thus, we can conclude that the binary logic $\mathbf{L}$ is Kripke complete with respect to the class of Kripke B-frames that satisfy $\forall x(xSx \rightarrow \forall y(ySx \rightarrow xSy))$.

Similarly, for all other cases we can show that each binary logic which is defined by adding to $\mathbf{L}_0$ the pair of formulas on the left hand side is Kripke complete with respect to the class of Kripke B-frames that satisfy the corresponding first order condition on the right hand side.

Note that, in (2) the pair $\langle p \wedge \boxminus p, \perp \rangle$ defines the *irreflexivity* of Kripke B-frames. This fact is in a sharp contrast to first order definability in modal logics, where the irreflexivity of a Kripke frame can not be defined by any modal formula.

Below, we rewrite every statement in Proposition 16 for normal modal logics by η_0 and ξ_0. Then we obtain the following.

(1^∂) $\mathcal{F} \models q \supset \Box\Diamond q$ if and only if $\mathcal{F} \models\!\!\!\equiv \forall x, y(yRx \rightarrow xRy)$.

(2^∂) $\mathcal{F} \models q \supset \Diamond q$ if and only if $\mathcal{F} \models\!\!\!\equiv \forall x(xRx)$.

(3^∂) $\mathcal{F} \models \Diamond q \supset \neg q$ if and only if $\mathcal{F} \models\!\!\!\equiv \forall x, y(\sim(xRy))$.

(4^∂) $\mathcal{F} \models \Box\neg q \supset \Box\Diamond q$ if and only if $\mathcal{F} \models\!\!\!\equiv \forall x, y, z(xRz \rightarrow ((xRy) or (zRy)))$.

(5^∂) $\mathcal{F} \models (q \wedge \Box\neg q) \supset \Box\Diamond q$ if and only if
$\mathcal{F} \models\!\!\!\equiv \forall x\{\exists y(xRy \,\&\, \sim(yRx)) \rightarrow xRx\}$.

(6^∂) $\mathcal{F} \models (\Box\neg q \wedge \Box\Diamond q) \supset \Box\neg\Box\Diamond q$ if and only if
$\mathcal{F} \models\!\!\!\equiv \forall x[\forall y\{xRy \rightarrow \exists z(yRz \,\&\, \sim(xRz))\} \rightarrow$
$\forall u\{xRu \rightarrow \exists v(\forall w(vRx \rightarrow xRw) \,\&\, uRv)\}]$.

For example, again for the case (5) and (5^∂), let $\mathbf{L} := \mathbf{L_0} + \{\langle p \wedge \boxminus p, \boxminus\boxminus p\rangle\}$. Then, since $\mathbf{L}$ is Kripke complete, by Theorem 14 and correspondences (5) in Proposition 16 and (5^∂), we obtain that $\mathbf{L}$ can be embedded into the normal modal logic $\mathbf{K} \oplus \{(q \wedge \Box\neg q) \supset \Box\Diamond q\}$ by η_0. (1^∂) and (2^∂) are well known as $\mathbf{B}$ and $\mathbf{T}$ respectively, whereas others are not familiar.

These facts are only a first step of correspondence theory for binary logics. To determine the class of formulas which characterizes some first order properties of Kripke B-frames will be an interesting problem. Also, to investigate the expressive power of formulas of binary logics by model theoretic arguments will be also another challenging project.

4 Goldblatt's translation of Ext(O) into Next(KTB)

In [2], Goldblatt used another translation of formulas to show that $\mathbf{O}$ is embeddable into $\mathbf{KTB}$, namely, η_1 defined as: $\eta_1(\perp) := \perp$, $\eta_1(p_i) := \Box\Diamond q_i$, $\eta_1(\alpha \wedge \beta) := \eta_1(\alpha) \wedge \eta_1(\beta)$, and $\eta_1(\boxminus\alpha) := \Box\neg\eta_1(\alpha)$.

According to Proposition 16 and the discussion following it, the binary logic $\mathbf{L_0} + \{\langle p, \boxminus\boxminus p\rangle, \langle p \wedge \boxminus p, \bot\rangle\}$ can also be embedded into the modal logic **KTB** by η_0. The difference between $\eta_0(p_i)$ and $\eta_1(p_i)$ arises from the axiom scheme $\langle\boxminus\boxminus p, p\rangle$ of orthologics. There seems to be no first order condition on Kripke B-frames to characterize this axiom scheme. In order to overcome this disadvantage, Goldblatt uses a little trick (see [2]).

In this section, we discuss the relation between $\mathrm{E}\textsc{xt}(\mathbf{O})$ and $\mathrm{N}\textsc{ext}(\mathbf{KTB})$ by the translation η_1 with the help of algebraic semantics and Kripke-type semantics. Since we have the *double negation law* $(\langle p, \boxminus\boxminus p\rangle, \langle\boxminus\boxminus p, p\rangle)$ in this case, we can go further to obtain finer results on embeddability of orthologics into normal modal logics over **KTB**, that are presented in the next section.

We begin with the relation between the algebraic semantics first. An *ortholattice* is an ABL $\mathfrak{A} = \langle \mathsf{A}, \cap, \cup, J, 0, 1\rangle$ in which the operation J satisfies the following conditions. For any $x \in \mathsf{A}$, (1): $J(J(x)) = x$, and (2): $x \cap J(x) = 0$. Note that in an ortholattice, a operation $\cup$ (join) can be defined as $x \cup y := J(J(x) \cap J(y))$ and so, an ortholattice is a lattice. An *orthologic* is a binary logic which contains **O**. Any orthologic is characterized by a class of ortholattices.

A **KTB**-algebra is a modal algebra $\mathfrak{B} = \langle \mathsf{B}, \cap, \cup, -, I, 0, 1\rangle$ in which the operation I satisfies (1): $I(x) \leq x$, and (2): $x \leq I(-I(-x))$. The modal logic **KTB** is characterized by the class of all **KTB**-algebras. The conditions (1) and (2) correspond to **T** and **B** respectively. We call a normal modal logic over **KTB** a **KTB**-*logic*.

For a **KTB**-algebra $\mathfrak{B}$, we can also define an algebra $\widehat{\mathfrak{A}_1}(\mathfrak{B}) := \langle \mathcal{R}(\mathsf{B}), \cap, \cup, J_{\mathfrak{B}}, 0, 1\rangle$, where $\mathcal{R}(\mathsf{B}) := \{a \in \mathsf{B} \mid I(-I(-a)) = a\}$ and $J_{\mathfrak{B}}(x) = I(-x)$. It is easily seen that this $\widehat{\mathfrak{A}_1}(\mathfrak{B})$ is an ortholattice for any **KTB**-algebra $\mathfrak{B}$. An element $a \in \mathcal{R}(\mathsf{B})$ is called *regular*. Now we have the following.

LEMMA 17. *For $\alpha, \beta \in \Phi$, $\widehat{\mathfrak{A}_1}(\mathfrak{B}) \models \langle\alpha, \beta\rangle$ if and only if $\mathfrak{B} \models \eta_1(\alpha) \supset \eta_1(\beta)$.*

Proof. This proof is basically the same as that for Lemma 4. Note that for a modal valuation $u : \Psi \to \mathfrak{B}$, if we define a valuation $v : \Phi \to \mathcal{R}(\mathsf{B})$ by $v(p_i) := u(\eta_1(p_i)) = u(\Box\Diamond q_i)$, then we can show by induction that $v(\alpha) = u(\eta_1(\alpha)) \in \mathcal{R}(\mathsf{B})$ for all $\alpha \in \Phi$. Therefore in our proof, when a modal valuation $u : \Psi \to \mathfrak{B}$ is given, v may be defined as $v(p_i) := u(\eta_1(p_i))$. On the other hand, when a valuation $v : \Phi \to \mathcal{R}(\mathsf{B})$ is given, it is enough to define u as: $u(q_i) := v(p_i)$, because $v(p_i) = I-(I-v(p_i)) = I-(I-u(q_i)) = u(\Box\Diamond q_i) = u(\eta_1(p_i))$. ∎

From the above lemma, similar results as in the general case of binary logics follow, that is, (1) a necessary and sufficient condition for an orthologic to be embeddable into a **KTB**-logic, (2) the uniqueness of the orthologic that is embeddable into a given **KTB**-logic, and (3) a necessary and sufficient condition for an orthologic to have the greatest **KTB**-logic in its modal companions. Their proofs are quite the same as those of Theorem 5, 6, and Proposition 7.

However, we can get more in the case of orthologics. First, for an ortholattice $\mathfrak{A} = \langle \mathsf{A}, \cap, \cup, J, 0, 1 \rangle$, a modal algebra $\widehat{\mathfrak{B}}(\mathfrak{A})$ is defined as a structure $\langle \mathcal{P}(F(\mathsf{A})), \cap, \cup, -, I_R, \emptyset, F(\mathsf{A}) \rangle$, where

(1) $F(\mathsf{A}) := \{F \subseteq \mathsf{A} \mid F \text{ is a proper filter on } \mathfrak{A}\}$,

(2) $\cap, \cup, -$ are the set-theoretic intersection, union, and complement respectively,

(3) R is a binary relation on $F(\mathsf{A})$ defined as: ${}_F R_G$ if and only if $J(a) \in F$ implies $a \notin G$ for any $a \in \mathsf{A}$, and

(4) $I_R(X) := \{F \in F(\mathsf{A}) \mid \forall G({}_F R_G \text{ implies } G \in X)\}$ for $X \in \mathcal{P}(F(\mathsf{A}))$.

It is easy to see that $\widehat{\mathfrak{B}}(\mathfrak{A})$ is a **KTB**-algebra. Define a map $\varphi : \mathsf{A} \to \mathcal{P}(F(\mathsf{A}))$ as: $\varphi(a) := \{F \in F(\mathsf{A}) \mid a \in F\}$. This map φ has the following property.

LEMMA 18. *For $a, b \in \mathsf{A}$,*

(1) $\varphi(J(a)) = I_R(-\varphi(a))$. *(3)* $\varphi(a \cap b) = \varphi(a) \cap \varphi(b)$.

(2) $\varphi(a) = I_R(-I_R(-\varphi(a)))$. *(4)* $a \leq b$ *if and only if* $\varphi(a) \subseteq \varphi(b)$.

This means that there is an isomorphic copy $\varphi(\mathfrak{A})$ in $\widehat{\mathfrak{B}}(\mathfrak{A})$. This construction is a modification of Jónsson-Tarski representation of modal algebras [4]. It can also be applied for general ABLs and modal algebras, but in that case, we cannot continue the following discussion.

For an ortholattice $\mathfrak{A}$ whose universe is A, the regular elements of $\widehat{\mathfrak{B}}(\mathfrak{A})$ is $\mathcal{R}(\widehat{\mathfrak{B}}(\mathfrak{A})) := \{X \in \mathcal{P}(F(\mathsf{A})) \mid I_R(-I_R(-X)) = X\}$. We put $\varphi(\mathsf{A}) := \{\varphi(a) \mid a \in \mathsf{A}\}$. If A is finite, then we have the following.

LEMMA 19. *Let $\mathfrak{A}$ be an ortholattice whose universe A is finite. Then $\varphi(\mathsf{A}) = \mathcal{R}(\widehat{\mathfrak{B}}(\mathfrak{A}))$.*

Proof. It is straightforward to show that $\varphi(\mathsf{A}) \subseteq \mathcal{R}(\mathfrak{B}(\mathfrak{A}))$ by Lemma 18 (2). To show the converse, suppose $X \in \mathcal{R}(\mathfrak{B}(\mathfrak{A}))$. If $X = F(\mathsf{A})$, then of course, $X = \varphi(1) \in \varphi(\mathsf{A})$. Therefore consider the case where $X \neq F(\mathsf{A})$.

Take an arbitrary $x \in F(\mathsf{A}) - X$, that is, $x \notin X = I_R(-I_R(-X))$. Then there exists $y \in F(\mathsf{A})$ such that xRy and $y \notin -I_R(-X)$. The latter is equivalent to that for any $z \in X$, there exists an element $a(z) \in \mathsf{A}$ such that $J(a(z)) \in y$ and $z \in \varphi(a(z))$. Since z is an arbitrary element of X, we obtain $X \subseteq \bigcup_{z \in X} \varphi(a(z))$. Because A is finite, X is also finite and so, X can be expressed as $\{z_i \mid 1 \le i \le n\}$ for some n. Therefore $X \subseteq \bigcup_{i=1}^{n} \varphi(a(z_i))$. Putting $a(x) := a(z_1) \cup \cdots \cup a(z_n)$, we have

$$X \subseteq \varphi(a(x)). \tag{$\star$}$$

On the other hand, since $J(a(z_i)) \in y$ for $i = 1, 2, \ldots, n$, $\bigcap_{i=1}^{n} J(a(z_i)) \in y$. Because $J(\bigcap_{i=1}^{n} J(a(z_i))) = \bigcup_{i=1}^{n} a(z_i) = a(x)$, if $a(x) \in x$, then $J(J(a(x))) \in x$ and $J(a(x)) \in y$. This is a contradiction since xRy. Thus $a(x) \notin x$, that is, $x \in -\varphi(a(x))$. Since x is an arbitrary element of $-X$, we obtain

$$-X \subseteq \bigcup_{x \in -X} -\varphi(a(x)) \tag{$\sharp$}$$

By $(\star)$, we have also $-X \supseteq -\varphi(a(x))$, and so, $-X \supseteq \bigcup_{x \in -X} -\varphi(a(x))$. This, together with $(\sharp)$ implies that $-X = \bigcup_{x \in -X} -\varphi(a(x))$. Since $-X$ is finite, and so $-X = \{x_j \mid 1 \le j \le m\}$ for some m, we obtain $-X = \bigcup_{j=1}^{m} -\varphi(a(x_j))$. Therefore $X = \bigcap_{j=1}^{m} \varphi(a(x_j)) = \varphi(\bigcap_{j=1}^{m} a(x_j))$. Thus $X \in \varphi(\mathsf{A})$. ■

This proof is based on representation of ortholattices by Goldblatt [3]. In the case of ABLs, we do not have a nice representation theorem for them, and therefore no similar result seems to be in sight. Now we can prove that for a finite ortholattice $\mathfrak{A}$, $\mathbf{L}(\mathfrak{A})$ has a modal companion by η_1.

LEMMA 20. *Let $\mathfrak{A}$ be any finite ortholattice. Then for any $\alpha, \beta \in \Phi$, $\mathfrak{A} \models \langle \alpha, \beta \rangle$ if and only if $\widehat{\mathfrak{B}}(\mathfrak{A}) \models \eta_1(\alpha) \supset \eta_1(\beta)$.*

Proof. The direction from right to left is trivial by Lemma 18. Suppose $\widehat{\mathfrak{B}}(\mathfrak{A}) \not\models \eta_1(\alpha) \supset \eta_1(\beta)$. Then there exists a modal valuation $u : \Psi \to \widehat{\mathfrak{B}}(\mathfrak{A})$ such that $u(\eta_1(\alpha)) \not\subseteq u(\eta_1(\beta))$. Since for every p_i, $u(\eta_1(p_i))$ is regular in $\widehat{\mathfrak{B}}(\mathfrak{A})$, there exists the unique element $a_i \in \mathsf{A}$ such that $u(\eta_1(p_i)) = \varphi(a_i)$

by Lemma 19. Now define a valuation $v : \Phi \to \mathfrak{A}$ by $v(p_i) := a_i$ for every p_i. Then, of course, $\varphi(v(p_i)) = u(\eta_1(p_i))$. It can be shown easily by induction that for any $\gamma \in \Phi$, $\varphi(v(\gamma)) = u(\eta(\gamma))$. Then by our supposition, we have $\varphi(v(\alpha)) \not\subseteq \varphi(v(\beta))$, which means that $v(\alpha) \not\leq v(\beta)$. Thus we have $\mathfrak{A} \not\models \langle\alpha, \beta\rangle$. ■

THEOREM 21.

(1) *Let $\mathfrak{A}$ be a finite ortholattice. Then, for any $\alpha, \beta \in \Phi$, $\langle\alpha, \beta\rangle \in \mathbf{L}(\mathfrak{A})$ if and only if $\eta(\alpha) \supset \eta(\beta) \in \mathbf{M}(\widehat{\mathfrak{B}}(\mathfrak{A}))$, that is, every tabular orthologic is embeddable into a tabular modal logic in* $\textsc{Next}(\mathbf{KTB})$.

(2) *Every orthologic that has the finite model property is embeddable by η_1 into a normal modal logic in* $\textsc{Next}(\mathbf{KTB})$*, which also has the finite model property.*

Next we deal with Kripke-type semantics of orthologics and **KTB**-logics, and the embeddability relation between these logics. An *orthoframe* is a Kripke B-frame $\mathcal{G} = \langle W, S\rangle$, where S is irreflexive and symmetric, and an *orthomodel* on $\mathcal{G}$ is a Kripke B-model $\mathfrak{N} = \langle\mathcal{G}, U\rangle$, where the valuation U fulfills the following restriction. For any variable p_i,

$$J_S(J_S(U(p_i))) = U(p_i) \qquad (*)$$

We use $\models^*$ for an orthomodel to express that its valuation satisfies $(*)$. Then the following holds.

THEOREM 22. *Let Ω be the class of all orthoframes. For any set of formulas Γ and any formula α, $\Omega : \Gamma \models^* \alpha$ if and only if $\Gamma \vdash_{\mathbf{O}} \alpha$.* □

Kripke frames which characterize the modal logic **KTB**, that is, *reflexive* and *symmetric* Kripke frames, are called **KTB**-Kripke frames. Kripke models on them are called **KTB**-Kripke models.

A pair of transformations ξ_1 and ξ_1^{-1} between orthomodels and **KTB**-Kripke models is defined as follows: For an orthomodel $\mathfrak{N} = \langle W, S, U\rangle$, a **KTB**-Kripke model $\xi_1(\mathfrak{N}) := \langle W, \check{S}, V\rangle$, where $V(q_i) := U(p_i)$. For a **KTB**-Kripke model $\mathfrak{M} = \langle W, R, V\rangle$, an orthomodel $\xi_1^{-1}(\mathfrak{M}) := \langle W, \check{R}, U\rangle$, where $U(p_i) := \{x \in W \mid \mathfrak{M} \models_x \Box\Diamond q_i\}$. ξ_1 and ξ_1^{-1} are mutually inverse in frame part, and we also have the following, quite similar to Lemma 13.

LEMMA 23.

(1) *Let $\varphi \in \Phi$ and $\mathfrak{N} = \langle W, S, U\rangle$ be an orthomodel. For any $x \in W$, $\mathfrak{N} \models^*_x \varphi$ if and only if $\xi_1(\mathfrak{N}) \models_x \eta_1(\varphi)$.*

(2) Let $\varphi \in \Phi$ and $\mathfrak{M} = \langle W, R, V\rangle$ be a **KTB***-Kripke model. For any $x \in W$, $\mathfrak{M} \models_x \eta_1(\varphi)$ if and only if $\xi_1^{-1}(\mathfrak{M}) \models^*_x \varphi$.*

By this lemma, we can prove analogue of Theorems 14 and 15. But for correspondence theory on orthoframes, things are more delicate because of the restriction $(*)$.

5 Two results on the embeddability of orthologics into normal logics over KTB

We finish off with two more results which are closely related to the embeddability relation between $\mathrm{E{\small XT}}(\mathbf{O})$ and $\mathrm{N{\small EXT}}(\mathbf{KTB})$ by η_1.

5.1 A continuum of normal modal logics over KTB

Recently the author proved [8] that there exists a continuum of orthologics by employing a class of ortholattices of special type. From this fact together with our embeddability of tabular orthologics, the existence of a continuum of **KTB**-logics follows. Our ortholattices are shown in the figure below. In this figure, the left and the right a_0's should be identified. So should the left and the right b_0's. We can see that the greatest element 1 is surrounded by $2n-1$ a_k's, and that the least element 0 is surrounded by $2n-1$ b_k's. The operation J is defined in the following way, so as to meet the requirement of definition for ortholattices.

$$J(a_0) := b_{n-1}, J(a_1) := b_n, \ldots\ldots, J(a_n) := b_0, \ldots\ldots, J(a_{2n-2}) := b_{n-2}$$

Of course, $J(1) = 0$ and $J(J(x)) = x$ for every element x. We name this type of ortholattices *chandelier type*, and denote the ortholattice of chandelier type with $2n-1$ atoms and $2n-1$ coatoms by $\mathfrak{A}_{2n-1}$. Its atoms and coatoms are denoted by b_k's and a_k's ($k = 0, 1, \ldots 2n-2$) respectively. The class of all ortholattices of chandelier type is denoted by $\mathcal{C}h$, more precisely, $\mathcal{C}h := \{\mathfrak{A}_{2k-1} \,|\, k \geq 3\}$. It is easily seen that no member of $\mathcal{C}h$ is orthomodular. To see that, notice that each member of $\mathcal{C}h$ has a subortholattice of benzene-type, and thus it is not orthomodular, by using the well-known criterion for orthomodular lattices [5]. This type of lattice was used by McKenzie [7] to prove that there exists a continuum of equational theories of lattices. The author modified his lattices to be ortholattices.

Let ℓ be any odd number greater than or equal to 5. For propositional variables $p_0, p_1, \ldots, p_{\ell-1}$, define two formulas α_ℓ, β_ℓ as follows:

$$\alpha_\ell := \bigwedge_{i \not\equiv j, i \not\equiv k, j \equiv k+1} p_i \vee (p_j \wedge p_k), \qquad \beta_\ell := \bigvee_{s \not\equiv t \pm 1} (p_s \wedge p_t),$$

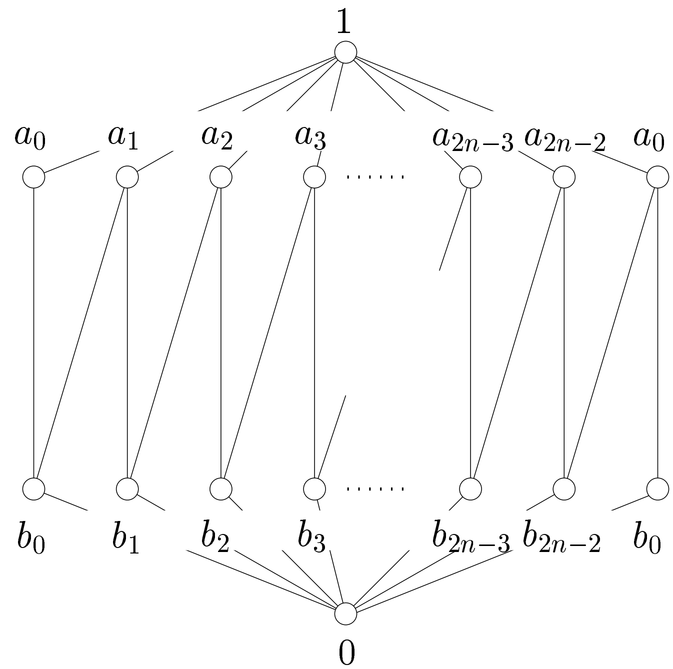

Figure 1. The ortholattice of chandelier type $\mathfrak{A}_{2n-1}$

where $\equiv$ expresses the congruence relation modulo ℓ. On these ortholattices and formulas we have the following [8].

LEMMA 24. *Let ℓ, m be odd numbers greater than or equal to* 5*.*

(1) $\mathfrak{A}_\ell \not\models \langle \alpha_\ell, \beta_\ell \rangle$.

(2) *For $m \neq \ell$,* $\mathfrak{A}_m \models \langle \alpha_\ell, \beta_\ell \rangle$.

By the above lemma, the following holds.

THEOREM 25.

(1) *For different subclasses $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{C}h$,* $\mathbf{L}(\mathcal{C}_1) \neq \mathbf{L}(\mathcal{C}_2)$.

(2) *There exists a continuum of orthologics.*

Proof. (1): We may assume that $\mathcal{C}_1 \not\subseteq \mathcal{C}_2$. Then there exists a number ℓ such that $\mathfrak{A}_\ell \in \mathcal{C}_1$ but $\mathfrak{A}_\ell \notin \mathcal{C}_2$. By Lemma 24, we have that $\mathcal{C}_1 \not\models \langle \alpha_\ell, \beta_\ell \rangle$ and that $\mathcal{C}_2 \models \langle \alpha_\ell, \beta_\ell \rangle$. This means that $\mathbf{L}(\mathcal{C}_2) \not\subseteq \mathbf{L}(\mathcal{C}_1)$.
(2): Since the cardinality of $\mathcal{C}h$ is $\aleph_0$, the cardinality of all subclasses of $\mathcal{C}h$ is $2^{\aleph_0}$, which is, by (1) above, the same cardinality of orthologics which are determined by some subclass of $\mathcal{C}h$. ∎

Because every member of $\mathcal{C}h$ is a finite ortholattice, we can show the following by Lemma 20.

LEMMA 26. *Let ℓ, m be odd numbers greater or equal to* 5. *Then,*

(1) $\mathfrak{B}(\mathfrak{A}_\ell) \not\models \eta_1(\alpha_\ell) \supset \eta_1(\beta_\ell)$.

(2) For $m \neq \ell$, $\mathfrak{B}(\mathfrak{A}_m) \models \eta_1(\alpha_\ell) \supset \eta_1(\beta_\ell)$.

Denote $\mathfrak{B}(\mathcal{C}h) := \{\mathfrak{B}(\mathfrak{A}_{2k-1}) \mid k \geq 3\}$. Then now we can show our result.

THEOREM 27. *For all $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathfrak{B}(\mathcal{C}h)$, if $\mathcal{D}_1 \neq \mathcal{D}_2$, then $\mathbf{M}(\mathcal{D}_1) \neq \mathbf{M}(\mathcal{D}_2)$. Therefore, there exists a continuum of* **KTB***-logics.*

5.2 Infinitely many modal companions of every tabular orthologic

We consider orthologics which are characterized by single finite orthoframes here. We show that every such orthologic has infinitely many modal companions. For an orthoframe $\mathcal{G}$, we denote $\mathbf{L}(\mathcal{G}) := \{\langle \alpha, \beta \rangle \mid \mathcal{G} \models \langle \alpha, \beta \rangle\}$. Similarly, for a **KTB**-Kripke frame $\mathcal{F}$, we denote $\mathbf{M}(\mathcal{F}) := \{A \mid \mathcal{F} \models A\}$. Then by Lemma 23, we have the following.

LEMMA 28. *Let $\mathcal{G}$ be an orthoframe and $\mathcal{F}$ a* **KTB***-Kripke frame. For any $\alpha, \beta \in \Phi$,*

(1) $\langle \alpha, \beta \rangle \in \mathbf{L}(\mathcal{G})$ *if and only if* $\eta_1(\alpha) \supset \eta_1(\beta) \in \mathbf{M}(\xi_1(\mathcal{G}))$.

(2) $\langle \alpha, \beta \rangle \in \mathbf{L}(\xi_1^{-1}(\mathcal{F}))$ *if and only if* $\eta_1(\alpha) \supset \eta_1(\beta) \in \mathbf{M}(\mathcal{F})$.

Let $\mathcal{G} = \langle W, S \rangle$ be an orthoframe in which $|W| = n$ ($n \in \omega$). For this $\mathcal{G}$, define $\mathcal{G}^{(+)} := \langle W \cup \{a\}, S \rangle$, where a is a new irreflexive point and nothing is changed except this. Then the following holds.

PROPOSITION 29.

(1) $\mathbf{L}(\mathcal{G}) = \mathbf{L}(\mathcal{G}^{(+)})$.

(2) There exists a formula $A \in \Psi$ such that $A \in \mathbf{M}(\xi_1^{-1}(\mathcal{G}))$ and that $A \notin \mathbf{M}(\xi_1^{-1}(\mathcal{G}^{(+)}))$

Proof. (1): This can be proved by using the fact that all regular subsets in W coincide with all regular subsets in $W \cup \{a\}$ since a is not related to any points in W by S.
(2): In $\xi_1^{-1}(\mathcal{G}^{(+)})$, a is the point that is related to every point (including a itself) by $\check{S}$, therefore a has $n+1$ *arms*, whereas $\xi_1^{-1}(\mathcal{G})$ has no such point because it has only n points. Consider formulas $F_0, F_1, \ldots, F_n \in \Psi$ such that each of them is satisfiable but that every couple of them cannot be satisfied at the same one point. In fact, for any $k \in \omega$, such k formulas can be effectively picked up. (For example, for $n = 3$, we can pick up $F_0 = q_1 \wedge q_2$, $F_1 = \neg q_1 \wedge q_2$, $F_2 = q_1 \wedge \neg q_2$, and $F_3 = \neg q_1 \wedge \neg q_2$.) Define $B := F_0 \wedge \Diamond F_1 \wedge \cdots \wedge \Diamond F_n$. For an appropriate valuation V, $\langle \xi_1^{-1}(\mathcal{G}^{(+)}), V \rangle \models_a B$, but there is no such point in $\xi_1^{-1}(\mathcal{G})$ that satisfies B. Therefore we have $\xi_1^{-1}(\mathcal{G}) \models \neg B$ and $\xi_1^{-1}(\mathcal{G}^{(+)}) \not\models \neg B$, which means that $\neg B \in \mathbf{M}(\xi_1^{-1}(\mathcal{G}))$ and that $\neg B \notin \mathbf{M}(\xi_1^{-1}(\mathcal{G}^{(+)}))$. ■

Now we have our result.

THEOREM 30. *Let $\mathcal{G}$ be a finite orthoframe. Then there exists an infinite descending chain of modal logics in* $\textsc{Next}(\mathbf{KTB})$ *into each of which $\mathbf{L}(\mathcal{G})$ can be embedded by η_1.*

Proof. Suppose $\mathcal{G}$ has n points and is denoted by $\mathcal{G}_{(n)}$. For each k, $\mathcal{G}_{(n+k)}$ denotes the orthoframe by only adding k irreflexive points to $\mathcal{G}_{(n)}$ with everything else untouched. Then by the previous Proposition, $\mathbf{L}(\mathcal{G}_{(n)}) = \mathbf{L}(\mathcal{G}_{(n+k)})$ for any k. Put $\mathbf{M}_\ell := \bigcap_{k=0}^{\ell} \mathbf{M}(\xi_1^{-1}(\mathcal{G}_{(n+k)}))$, then trivially, $\mathbf{M}_0 \supseteq \mathbf{M}_1 \supseteq \cdots$. Moreover, by a similar argument as in the proof of the proposition above, for every ℓ, there exists a formula $A_{\ell+1}$ such that $A_{\ell+1} \in \mathbf{M}_\ell$ and that $A_{\ell+1} \notin \mathbf{M}_{\ell+1}$. Thus $\mathbf{M}_\ell \supsetneq \mathbf{M}_{\ell+1}$. Finally, by Lemma 28, $\langle \alpha, \beta \rangle \in \mathbf{L}(\mathcal{G}_{(n)})$ if and only if $\eta_1(\alpha) \supset \eta_1(\beta) \in \mathbf{M}_\ell$ for any ℓ. ■

This work is only a first step of studying the relation between $\textsc{Ext}(\mathbf{L}_0)$ and $\textsc{Next}(\mathbf{K})$, and that between $\textsc{Ext}(\mathbf{O})$ and $\textsc{Next}(\mathbf{KTB})$. We have not yet reached complete answers to problems (1) and (2) in the first section. But we hope this study will give a framework in which the structure of the lattices of these classes of logics are clarified.

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