21

An Incompleteness Result for Predicate Extensions of Intermediate Propositional Logics

D. Skvortsov

ABSTRACT. For a continuum of predicate extensions of intermediate propositional logics of small height or of small width we establish incompleteness in the semantics of Kripke bundles introduced in [7].

The semantics KB of *Kripke bundles* was introduced in [7]. It generalizes the standard Kripke semantics for predicate superintuitionistic logics. Its further strengthening, the *functor semantics* (the semantics of SET-valued functors over categories, or so-called C*-sets*) is considered in S. Ghilardi's paper [2]. An essential point in studying different semantics is the comparison of their strength.

This can be done in two ways. First, we can compare the completeness of logics: in a stronger semantics, more logics are complete. Second, we can compare separability of certain formulas from certain logics. Clearly, if a formula A is unseparable from a logic $\mathbf{L}$, then every extension $\mathbf{L}'$ of $\mathbf{L}$ which does not contain A is incomplete. Most of incompleteness results are proved in this way.

In this paper we establish KB-unseparability of some formulas introduced in Ghilardi's paper [3] (these formulas describe, in some sense, binary trees of finite height) from superintuitionistic predicate logics of small height or small width. This implies, in particular, the KB-incompleteness of the predicate version **Q-L** of any intermediate propositional logic $\mathbf{L}$ of small height or width, and the incompleteness of many its extensions (with the same propositional fragment).

Note that the predicate version **Q-L** of a *canonical* propositional logic $\mathbf{L}$ is complete in functor semantics [4]. On the other hand, for any propositional logic $\mathbf{L}$ of finite height (except for classical logic) its predicate version **Q-L** is incomplete in standard Kripke semantics [6].

REMARK 1. The result presented in this paper was obtained almost ten years ago, in the early 1990s and was motivated by Ghilardi's papers [2, 3] and by constructions presented there. At that time it was probably the first

Advances in Modal Logic, Volume 4, 461–474.

explicit example of KB-incompleteness; an indirect KB-incompleteness proof based on non-axiomatizability of KB-completions of some finitely axiomatizable logics was sketched in [8].

In 1997, K. Nagaoka and E. Isoda [5] proved very strong results which imply the KB-incompleteness of predicate versions **Q-L** for all propositional logics **L** of finite height or of finite width except classical logic and Dummett's logic (in the latter two cases **Q-L** are Kripke-complete). However, the idea of the proof and constructions presented in [5] essentially differs from ours. Let us also point out that the KB-unseparability is proved in [5] for another kind of formulas, different from Ghilardi's ones. Thus the ranges of KB-incompleteness are definitely incomparable. For example, our construction can be used to show the KB-incompleteness of $[\mathbf{Q\text{-}L} + Z]$, where $Z = \forall x, y\ (P(x) \supset \neg\neg P(y))$, for some logics of finite width, while the C-set constructed in [5] (in its part referring to the width) refutes the formula Z.

Thus we present this paper almost in its original form, only with some minimal re-compositions and re-formulations.

In Section 1 we give necessary definitions and formulate the Main Theorem on KB-unseparability (and its consequences on KB-incompleteness). In Section 2 we formulate Technical Lemma and prove the Main Theorem basing on this Lemma. Finally, in Section 3 we prove this Technical Lemma.

1 Formulation of KB-incompleteness theorem

1.1 *Predicate superintuitionistic logics* are understood as usual (cf. e.g. [7, 1]), i.e., as sets of predicate formulas (without equality and function symbols), containing all axioms of Heyting predicate calculus and closed under modus ponens, generalization, and substitution of arbitrary formulas instead of atomic ones.

The *propositional fragment* of a predicate logic is the set of all its propositional formulas.

The *predicate version* **Q-L** of a propositional (intermediate) logic **L** is the minimal predicate logic, the propositional fragment of which equals to **L**. In particular, Heyting predicate logic **Q-H** is the predicate version of Heyting propositional logic **H**.

Let us consider propositional formulas Q_n and P_n for any $n \in \omega$:

$$Q_n = \bigvee_{i=0}^{n} [q_i \supset \bigvee_{j \neq i} q_j]$$

(in particular, $Q_0 = \neg q_0$, and $Q_1 = (q_0 \supset q_1) \vee (q_1 \supset q_0)$,
i.e., $(\mathbf{H} + Q_1)$ is the propositional Dummett's logic),

$$P_0 = \bot, P_{n+1} = q_n \vee (q_n \supset P_n),$$

$q_0, q_1, q_2, \ldots$ being a list of different propositional symbols.

Let also

$$J = \neg q_0 \vee \neg\neg q_0.$$

A propositional intermediate logic $\mathbf{L}$ is *of width* n if $Q_n \in \mathbf{L}, Q_{n-1} \notin \mathbf{L}$; a logic $\mathbf{L}$ is *of height* n if $P_n \in \mathbf{L}, P_{n-1} \notin \mathbf{L}$.

It is well-known that $\mathbf{L}$ is a logic of finite height iff $\mathbf{L} \not\subseteq (\mathbf{H} + Q_1)$.

Let $(W, \leq)$ be a pre-ordered set, i.e., $\leq$ is a reflexive and transitive, but not necessarily antisymmetric relation on W. As usual, the *propositional logic* $\mathbf{PL}(W, \leq)$ *of* $(W, \leq)$ is defined as the logic of the Heyting algebra of all $\leq$-stable (upward closed) subsets of W. It is well-known that if $(W, \leq)$ has the least element, then $\mathbf{PL}(W, \leq)$ is of width (or of height) $\leq n$ iff W has no antichains (or respectively, $<$-chains) of more than n elements. On the other hand, every propositional logic $\mathbf{L}$ of height $\leq n$ can be presented as $\mathbf{PL}(\Omega) = \bigcap(\mathbf{PL}(W, \leq) : (W, \leq) \in \Omega)$ for a family Ω of pre-ordered sets of height $\leq n$, i.e., of pre-ordered sets without $(n+1)$-element chains.

1.2 A *Kripke bundle* [7, Sect. 5] and [1] is a triple $\mathbb{F} = \langle W, D, \pi \rangle$ formed by non-empty pre-ordered sets W and D (R and ϱ denote corresponding pre-orders), and by a p-morphism π. This means that $\pi : D \to W$ is a surjective map such that

$$(i) a\varrho b \Rightarrow \pi(a) R \pi(b), \text{ and } (ii) \pi(a) R v \Rightarrow \exists b[(a\varrho b) \& (\pi(b) = v)].$$

We set:

$$D_u = \pi^{-1}(u) \text{ for } u \in W;$$

thus $D = \bigcup\limits_{u \in W} D_u, D_u \neq \varnothing$ for each $u \in W$, and $D_u \cap D_v = \varnothing$ if $u \neq v$.

We also set $D_n = \bigcup\limits_{u \in W} (D_u)^n$ for $n > 0$. D_n is pre-ordered by the relation:

$$(a_1, \ldots, a_n) \leq_n (b_1, \ldots, b_n) \Leftrightarrow \bigwedge_i (a_i \varrho b_i) \& \bigwedge_{i \neq j} [(a_i = a_j) \Rightarrow (b_i = b_j)].$$

In particular, $(D_1, \leq_1) = (D, \varrho)$. Finally we set: $(D_0, \leq_0) = (W, R)$.

Note that:

(π) for any $a_0, a_1, \ldots, a_n \in D_u$ and $b_1, \ldots, b_n \in D_v (n \geq 0)$
such that $(a_1, \ldots, a_n) \leq_n (b_1, \ldots, b_n)$ (hence $u \leq_0 v$)
there exists $b_0 \in D_v$ such that $(a_0, a_1, \ldots, a_n) \leq_{n+1} (b_0, b_1, \ldots, b_n)$

(cf. (ii)).

A *valuation* ξ in a bundle $\mathbb{F}$ (see [7]) can be defined as the forcing relation: $u \vDash A(\mathbf{a})$ for predicate formulas $A(x_1, \dots, x_n)$, $u \in W$, $\mathbf{a} = (a_1, \dots, a_n) \in (D_u)^n$, satisfying the *monotonicity*:

$$(u \leq_0 v) \& (\mathbf{a} \in (D_u)^n) \& (\mathbf{b} \in (D_v)^n) \& (\mathbf{a} \leq_n \mathbf{b}) \& (u \vDash A(\mathbf{a})) \Rightarrow (v \vDash A(\mathbf{b}))$$

and the following inductive clauses:

$$\begin{array}{l}
u \nvDash \bot; u \vDash (B \wedge C) \Leftrightarrow (u \vDash B \text{ and } u \vDash C);\\
u \vDash (B \vee C) \Leftrightarrow (u \vDash B \text{ or } u \vDash C);\\
u \vDash (B \supset C)(\mathbf{a}) \Leftrightarrow \forall v \geq u \forall \mathbf{b} \in (D_v)^n [(\mathbf{a} \leq_n \mathbf{b}) \& (v \vDash B(\mathbf{b})) \Rightarrow (v \vDash C(\mathbf{b}))];\\
u \vDash \forall y B(\mathbf{a}, y) \Leftrightarrow \forall v \geq u \forall \mathbf{b} \in (D_v)^n \forall c \in D_v [(\mathbf{a} \leq_n \mathbf{b}) \Rightarrow (v \vDash B(\mathbf{b}, c))];\\
u \vDash \exists y B(\mathbf{a}, y) \Leftrightarrow \exists c \in D_u (u \vDash B(\mathbf{a}, c)).
\end{array}$$

As usual, to obtain a valuation, it is sufficient to know $u \vDash A(\mathbf{a})$ only for atomic A, and the monotonicity for atomic A implies the monotonicity for arbitrary A, by induction.

A predicate formula $A(x_1, \dots, x_n)$ is called *valid* in a Kripke bundle $\mathbb{F}$ if $u \vDash A(\mathbf{a})$ for any valuation in $\mathbb{F}$, for any $u \in W$, and any $\mathbf{a} \in (D_u)^n$. A formula A is called *strongly valid* in $\mathbb{F}$ if all its substitution instances are valid in $\mathbb{F}$. The set $\mathbf{QL}(\mathbb{F})$ of all predicate formulas strongly valid in $\mathbb{F}$ is a predicate logic [7, Sect. 5]. A predicate logic $\mathbf{L}$ is called *KB-complete* if it can be presented as $\mathbf{L} = \bigcap(\mathbf{QL}(\mathbb{F}) : \mathbb{F} \in \Omega)$ for some family Ω of Kripke bundles.

The following simple lemma is analogous to Lemma 12 from [4] (its proof is postponed until Section 3):

LEMMA 2. *For any Kripke bundle $\mathbb{F}$, the propositional fragment of $\mathbf{QL}(\mathbb{F})$ is $\bigcap\limits_{n \in \omega} \mathbf{PL}(D_n, \leq_n)$.*

1.3 Let $T_h^2 = \{(d_1, \dots, d_k) : 0 \leq k \leq h, \forall i (d_i \in \{0, 1\})\}$ be the binary tree of height $(h+1)$ (for $h \in \omega, h > 0$). Recall the definition of the predicate formula $F_{\mathbf{d}}^h$ corresponding to an arbitrary $\mathbf{d} \in T_h^2$ given in [3, Sect. 5]. This is a formula with two monadic predicate symbols R_0, R_1 and with k parameters $x_1, \dots, x_k$ (where k is the length of $\mathbf{d}$):

$$\begin{array}{l}
F_{\mathbf{d}}^h = \bigvee\limits_{i=1}^{h} R_{d_i}(x_i) \text{ if } k = h,\\
F_{\mathbf{d}}^h = \forall x_{k+1} [(R_0(x_{k+1}) \supset F_{(\mathbf{d},1)}^h) \vee (R_1(x_{k+1}) \supset F_{(\mathbf{d},0)}^h)] \text{ if } k < h.
\end{array}$$

In [3] it is proved that $\mathbf{Q}\text{-}\mathbf{L} \nvdash \neg\neg F_\varepsilon^h$ for any propositional logic $\mathbf{L}$ such that $J \in \mathbf{L}$, $Q_1 \notin \mathbf{L}$, and for any $h > 0$, where ε is the empty sequence from T_h^2. More precisely, [3] describes a C-set separating $\neg\neg F_\varepsilon^h$ from **Q-L**. Actually the same C-set separates $\neg\neg F_\varepsilon^h$ from $[\mathbf{Q\text{-}L} + E + Z]$, where

$$E = \neg\neg\exists x P(x) \supset \exists x\ \neg\neg P(x),$$
$$Z = \forall x, y\ (P(x) \supset \neg\neg P(y)),$$

and from many other extensions of **Q-L**.

THEOREM 3 (on KB-unseparability). *Let $\mathbb{F}$ be a Kripke bundle such that $(J \wedge P_m) \in \mathbf{QL}(\mathbb{F})$ for some $m \in \omega$, and $\neg\neg F_\varepsilon^3 \notin \mathbf{QL}(\mathbb{F})$. Then the propositional fragment of $\mathbf{QL}(\mathbb{F})$ is a logic of height ≥ 5 and of width ≥ 5.*

In other words, if the formula $\neg\neg F_\varepsilon^3$ is KB-separable from a predicate logic $\mathbf{L}$ containing $J \wedge P_m$, then the propositional fragment of $\mathbf{L}$ has height ≥ 5 and width ≥ 5.

COROLLARY 4. *Let $\mathbf{L}$ be a propositional logic of finite height, $J \in \mathbf{L}$, $Q_1 \notin \mathbf{L}$. If the predicate version* **Q-L** is *KB-complete, then $\mathbf{L}$ is a logic of height ≥ 5 and of width ≥ 5.*

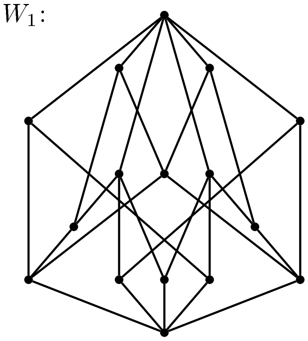
Hence the KB-incompleteness of **Q-L** follows for continuum of logics $\mathbf{L}$ of height 4, for $\mathbf{L} = (\mathbf{H} + P_k \wedge J)$, where $3 \leq k \leq 4$, and for $\mathbf{L} = (\mathbf{H} + P_n \wedge Q_m \wedge J)$, where $(2 \leq m \leq 4, n \geq 3)$ or $(3 \leq n \leq 4, m \geq 2)$. Similarly we obtain the KB-incompleteness of the logics $[\mathbf{Q\text{-}L} + E]$, $[\mathbf{Q\text{-}L} + Z]$, $[\mathbf{Q\text{-}L} + E + Z]$ (for the same $\mathbf{L}$), etc.

The bound 5 for height and width in Theorem on KB-unseparability cannot be increased, and we cannot say anything about the width without some additional restrictions referring to height. In fact, let us consider the p.o. sets W_1 and W_2 as in Fig.1: W_1 is of width and height 5, and W_2 is of width 2 (and of infinite height). For $s = 1, 2$ there exist Kripke bundles $\mathbb{F}_s$ (Fig.2) such that the propositional fragment of $\mathbf{QL}(\mathbb{F}_s)$ is $\mathbf{PL}(W_s)$ and $\neg\neg F_\varepsilon^h \notin \mathbf{QL}(\mathbb{F}_s)$ for all $h > 0$. In other words, all formulas $\neg\neg F_\varepsilon^h$ are separable from **Q-L** in the semantics of Kripke bundles KB if $\mathbf{L} = \mathbf{PL}(W_s), s = 1, 2$ (here $J \in \mathbf{L}, Q_1 \notin \mathbf{L}$). The proof for $\mathbb{F}_1$ is analogous to that of Theorem 5.6 in [3], the proof for $\mathbb{F}_2$ is similar.

If the formula $\neg\neg F_\varepsilon^3$ is replaced with a simpler formula $\neg\neg F_\varepsilon^2$, then the bound becomes 4 for height and 3 for width. The corresponding statement can be proved similarly. This result is also the optimal, because $\neg\neg F_\varepsilon^2$ is KB-separable from **Q-L** if $\mathbf{L} = \mathbf{PL}(W_3)$ (Fig.3). A corresponding bundle $\mathbb{F}_3$ is more complicated than $\mathbb{F}_1$, but it can be constructed in a straightforward way using the formulation of Technical Lemma (Lemma 5 below) for the case $h = 2$.

As for $h = 1$, the formula

$$\neg\neg F_\varepsilon^1 = \neg\neg\forall x[(R_0(x) \supset R_1(x)) \vee (R_1(x) \supset R_0(x))]$$



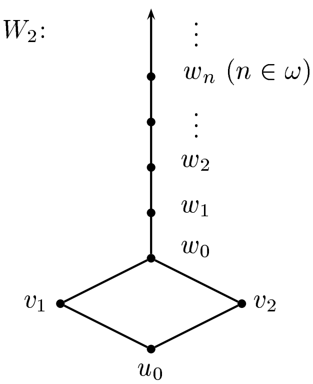


Figure 1.

is KB-separable from **Q-L** whenever $Q_1 \notin \mathbf{L}$; Fig. 4 shows the corresponding bundles.

If $\mathbf{L} \supseteq (\mathbf{H} + Q_1)$, then **Q-L** contains F^1_ε, leave alone $\neg\neg F^1_\varepsilon$.

Finally let us point out that the condition '$\exists m(P_m \in \mathbf{QL}(\mathbb{F}))$' in the statement of Theorem actually can be weakened to '$\mathbf{PL}(\mathbb{F}) \not\subseteq \mathbf{PL}(W_2)$', where $\mathbf{PL}(\mathbb{F})$ denotes the propositional fragment of $\mathbf{QL}(\mathbb{F})$.

2 Proof of KB-unseparability theorem

2.1 We use the following notation. For $\mathbf{d} = (d_1, \dots, d_k) \in T^2_h$, where $0 \leq k \leq h$, we put: $k^* = k+1$ if $k < h$, and $k^* = h$ if $k = h$, $\mathbf{d}_i = (d_1, \dots, d_i)$ for $1 \leq i \leq k$, $\mathbf{d}_0 = \varepsilon$, and $\mathbf{d}^- = \mathbf{d}_{k-1} = (d_1, \dots, d_{k-1})$ if $0 < k \leq h$ ($\mathbf{d}^-$ for $\mathbf{d} = \varepsilon$ is not defined).

A routine proof of the following lemma will be given in Section 3.

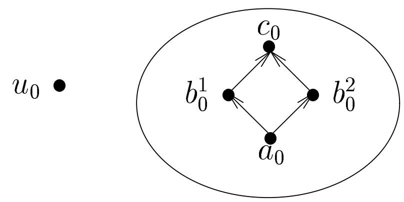
LEMMA 5 (Technical Lemma). *Let $\mathbb{F} = \langle W, D, \pi \rangle$ be a Kripke bundle such that $(J \wedge P_m) \in \mathbf{QL}(\mathbb{F})$ for some $m \in \omega$. Then the following conditions are equivalent (for any $h > 0$):*

(i) $\neg\neg F^h_\varepsilon \notin \mathbf{QL}(\mathbb{F})$;

(ii) there exists $n \in \omega$, and for all $\mathbf{d} = (d_1, \dots, d_k) \in T^2_h$ there exist $u^{\mathbf{d}} \in W, \mathbf{c}^{\mathbf{d}} = (c^{\mathbf{d}}_1, \dots, c^{\mathbf{d}}_n) \in (D_{u^{\mathbf{d}}})^n, \mathbf{a}^{\mathbf{d}} = (a^{\mathbf{d}}_1, \dots, a^{\mathbf{d}}_{k^*}) \in (D_{u^{\mathbf{d}}})^{k^*}$, such that

(0) $\forall \mathbf{c} \in D_n(\mathbf{c}^\varepsilon \leq_n \mathbf{c} \Rightarrow \mathbf{c} \leq_n \mathbf{c}^\varepsilon)$
(and thus $\forall v \in W(u^\varepsilon \leq_0 v \Rightarrow v \leq_0 u^\varepsilon)$);

$\mathbb{F}_1: \quad W = \{u_0\} \qquad D = D_{u_0}$



$\mathbb{F}_2: \quad W = \{u_n : n \in \omega\}$

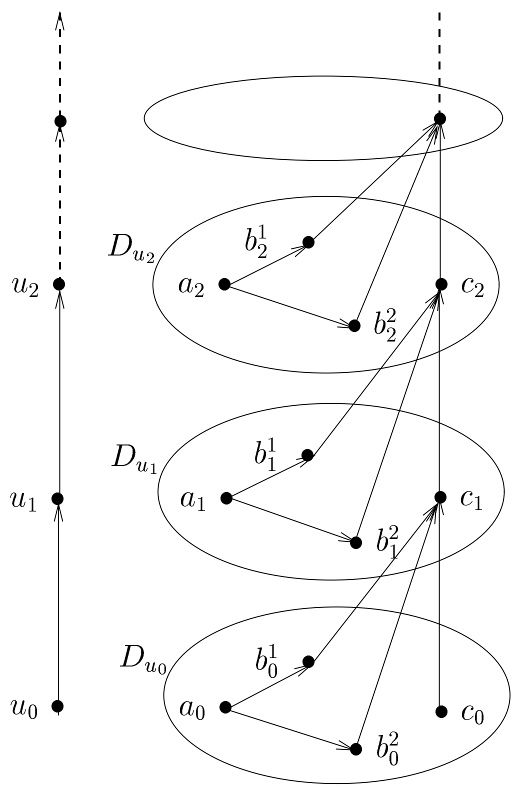


Figure 2.

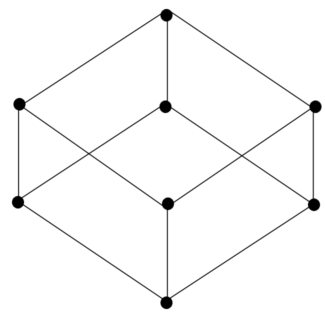
W_3: 

Figure 3.

(1) $(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-}) \leq_{n+k} (\mathbf{c}^{\mathbf{d}}, (\mathbf{a}^{\mathbf{d}})^-)$ *for* $\mathbf{d} = (d_1, \dots, d_k), 0 < k \leq h$, *where* $(\mathbf{a}^{\mathbf{d}})^- = (a_1^{\mathbf{d}}, \dots, a_k^{\mathbf{d}})$ *(and thus* $u^\varepsilon \leq_0 u^{\mathbf{d}}, \mathbf{c}^\varepsilon \leq_n \mathbf{c}^{\mathbf{d}}$ *for any* $\mathbf{d} \in T_h^2$*);*

(2) $(\mathbf{c}^{\mathbf{d}}, a_i^{\mathbf{d}}) \not\leq_{n+1} (\mathbf{c}^{\mathbf{d}'}, a_j^{\mathbf{d}'})$ *if* $\mathbf{d} = (d_1, \dots, d_i, \dots, d_k)$, $\mathbf{d}' = (d_1', \dots, d_j', \dots, d_{k'}'), d_i \neq d_j'$ *(here* $1 \leq i \leq k \leq h, 1 \leq j \leq k' \leq h$*);*

(Warning: $u^{\mathbf{d}}, a_i^{\mathbf{d}}, c_i^{\mathbf{d}}$ *are not necessarily different)*

(iii) there exist $n \in \omega$, $u^{\mathbf{d}} \in W, \mathbf{c}^{\mathbf{d}} \in (D_{u^{\mathbf{d}}})^n, \mathbf{a}^{\mathbf{d}} \in (D_{u^{\mathbf{d}}})^{k^*}$ *as in (ii), and also* $b^{\mathbf{d}} \in D_{u^\varepsilon}$ *for all* $\mathbf{d} = (d_1, \dots, d_k) \in T_{h-1}^2, 0 \leq k < h$*, such that*

(3) $(\mathbf{c}^\varepsilon, b^{\mathbf{d}}) \not\leq_{n+1} (\mathbf{c}^{\mathbf{d}'}, a_i^{\mathbf{d}'})$ *for all* $\mathbf{d}' = (d_1', \dots, d_j') \in T_h^2, 1 \leq i \leq j \leq h$*;*

(4) $(\mathbf{c}^{\mathbf{d}'}, a_1^{\mathbf{d}'}, \dots, a_{k+1}^{\mathbf{d}'}) \leq_{n+k+1} (\mathbf{c}^\varepsilon, b^{\mathbf{d}_0}, \dots, b^{\mathbf{d}_k})$ *for any* $\mathbf{d}' = (d_1, \dots, d_k, d_{k+1}', \dots, d_h')$.

2.2 Proof of KB-unseparability theorem We use Technical Lemma for $h = 3$. Let $u = u^{(0,1)}, (\mathbf{c}, a_1, a_2, a_3) = (\mathbf{c}^{(0,1)}, \mathbf{a}^{(0,1)}) \in (D_u)^{n+3}, u^d = u^{(0,1,d)}$, $(\mathbf{c}^d, a_1^d, a_2^d, a_3^d) = (\mathbf{c}^{(0,1,d)}, \mathbf{a}^{(0,1,d)}) \in (D_{u^d})^{n+3}$, for $d \in \{0,1\}$. Let $\mathbf{c} = (c_1, \dots, c_n), \mathbf{c}^d = (c_1^d, \dots, c_n^d)$. Take $b_i \in D_u, b_i^d \in D_{u^d}$ for $i \in \{1,2,3\}, d \in \{0,1\}$, such that

$$(\mathbf{c}^\varepsilon, b^\varepsilon, b^{(0)}, b^{(0,1)}) \leq_{n+3} (\mathbf{c}, b_1, b_2, b_3) \leq_{n+3} (\mathbf{c}^d, b_1^d, b_2^d, b_3^d)$$

for $d \in \{0,1\}$. Then $(\mathbf{c}, a_1, a_2, a_3) \leq_{n+3} (\mathbf{c}^d, a_1^d, a_2^d, a_3^d) \leq_{n+3} (\mathbf{c}, b_1, b_2, b_3)$ for $d \in \{0,1\}$ and $(\mathbf{c}', a') \not\leq_{n+1} (\mathbf{c}'', a''), (\mathbf{c}'', a'') \not\leq_{n+1} (\mathbf{c}', a')$ for any

$$(\mathbf{c}', a') \in \{(\mathbf{c}, a_1), (\mathbf{c}^0, a_1^0), (\mathbf{c}^1, a_1^1), (\mathbf{c}^0, a_3^0)\},$$

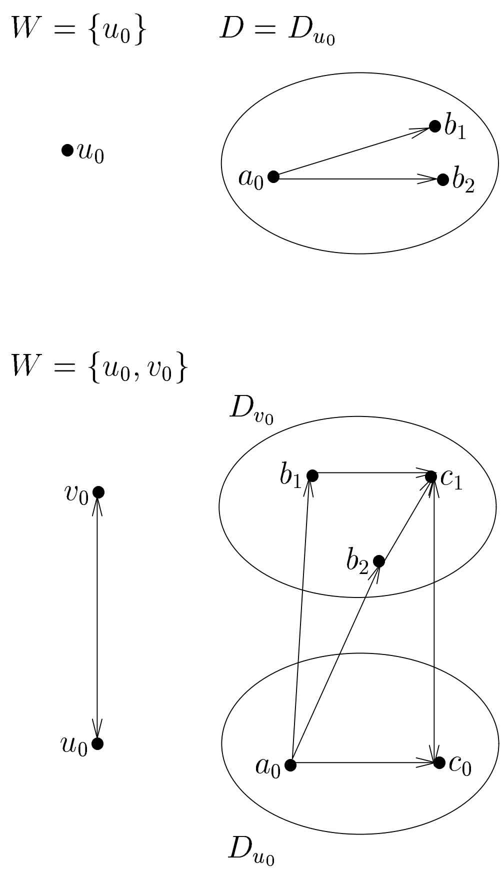


Figure 4.

$$(\mathbf{c}'', a'') \in \{(\mathbf{c}, a_2), (\mathbf{c}^0, a_2^0), (\mathbf{c}^1, a_2^1), (\mathbf{c}^1, a_3^1)\}.$$

Also $(\mathbf{c}, b_i) \not\leq_{n+1} (\mathbf{c}^d, a_j^d)$ for any $i, j \in \{1,2,3\}, d \in \{0,1\}$ (see Fig.5, where directed arrows correspond to $<$, dashed lines in the upward direction correspond to $\leq$, and different signs near a^i, a_i^d mean that the corresponding $(\mathbf{c}, a_i), (\mathbf{c}^d, a_i^d)$ are incomparable).

So it follows that $a_i \notin \{c_1, \ldots, c_n, b_1, b_2, b_3\}$ for $i \in \{1,2,3\}$ — since $(\mathbf{c}, a_i) \leq_{n+1} (\mathbf{c}, b_i)$ and $(\mathbf{c}, b_i) \not\leq_{n+1} (\mathbf{c}, a_j)$ for $i, j \in \{1,2,3\}$. Similarly we have $a_i^d \notin \{c_1^d, \ldots, c_n^d, b_1^d, b_2^d, b_3^d\}$ for $d \in \{0,1\}, i \in \{1,2,3\}$. Also $a_i \neq a_j$ for any $i \neq j (\in \{1,2,3\})$ — since $(\mathbf{c}, a_1) \not\leq_{n+1} (\mathbf{c}, a_2), (\mathbf{c}, a_1) \not\leq_{n+1} (\mathbf{c}^1, a_3^1), (\mathbf{c}, a_2) \not\leq_{n+1} (\mathbf{c}^0, a_3^0)$, and thus $(\mathbf{c}, a_1) \not\leq_{n+1} (\mathbf{c}, a_3), (\mathbf{c}, a_2) \not\leq_{n+1}$

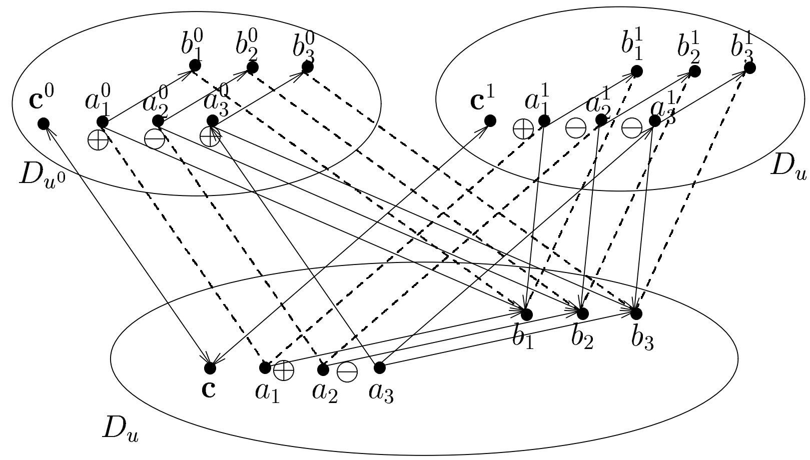


Figure 5.

$(\mathbf{c}, a_3)$.

We obtain that $(\mathbf{c}, a_1, a_2, a_3) <_{n+3} (\mathbf{c}, b_1, a_2, a_3) <_{n+3} (\mathbf{c}, b_1, b_2, a_3) <_{n+3}$ $<_{n+3}$ $(\mathbf{c}^0, b_1^0, b_2^0, a_3^0)$ $<_{n+3}$ $(\mathbf{c}^0, b_1^0, b_2^0, b_3^0)$, taking into account that $(\mathbf{c}^0, a_3^0)$
$\not\leq_{n+1} (\mathbf{c}, a_3)$ since $(\mathbf{c}^0, a_3^0) \not\leq_{n+1} (\mathbf{c}^1, a_3^1)$. Hence $\mathbf{PL}(D_{n+3}, \leq_{n+3})$ is a logic of height ≥ 5.

Also $(\mathbf{c}, a_1, a_2, a_3) \leq_{n+3} (\mathbf{c}', \mathbf{a}')$ for any $(\mathbf{c}', \mathbf{a}')$ from

$$X = \{(\mathbf{c}^0, a_1^0, a_2^0, b_3^0), (\mathbf{c}^0, a_1^0, b_2^0, a_3^0), (\mathbf{c}^0, b_1^0, a_2^0, a_3^0), (\mathbf{c}^1, a_1^1, b_2^1, a_3^1), (\mathbf{c}^1, b_1^1, a_2^1, a_3^1)\},$$

and X is a $\leq_{n+3}$-antichain in D_{n+3} (since $(\mathbf{c}^0, a_3^0) \not\leq_{n+1} (\mathbf{c}^1, a_3^1), (\mathbf{c}^1, a_3^1)$ $\not\leq_{n+1} (\mathbf{c}^0, a_3^0)$, and $(\mathbf{c}^d, b_i^d) \not\leq_{n+1} (\mathbf{c}^e, a_i^e)$ for any $d, e \in \{0, 1\}, i \in \{1, 2, 3\}$). Hence $\mathbf{PL}(D_{n+3}, \leq_{n+3})$ is a logic of width ≥ 5. ■

Note that Technical Lemma actually gives more information on Kripke bundles that refute the formulas $\neg\neg F_\varepsilon^h$ (for $h > 0$) than we need for the proof of our theorem. Thus one can obtain some versions of KB-unseparability theorem using other parameters than height or width. There are also versions formulated in terms of 'embeddings' in frames of certain logics.

3 Proof of technical lemma

3.1 Let A be a predicate formula, $(P_1^{(n_1)}, \ldots, P_m^{(n_m)})$ be the list of predicate symbols occurring in A, and $(z_1, \ldots, z_n), (Q_1^{(n+n_1)}, \ldots, Q_m^{(n+n_m)})$ be lists of different individual variables and predicate symbols non-occurring in A. Then $A^{[n]}$ is a substitution instance of A obtained by replacing every occurrence of $P_i(y_1,\dots,y_{n_i})$ in A with $Q_i(z_1,\dots,z_n,y_1,\dots,y_{n_i})$.

LEMMA 6. *A formula A is strongly valid in a bundle $\mathbb{F}$ iff all $A^{[n]}$ are valid in $\mathbb{F}$ for $n \in \omega$.*

Proof. Let B be a substitution instance of A obtained by replacing every occurrence of $P_i(y_1,\dots,y_{n_i})$ in A with $B_i(z_1,\dots,z_n,y_1,\dots,y_{n_i})$ $(1 \leq i \leq m)$ and $(x_1,\dots,x_k),(z_1,\dots,z_n,x_1,\dots,x_k)$ be lists of free variables of formulas A and B respectively. Let ξ be a valuation in $\mathbb{F}$ such that $u \not\models B(c_1,\dots,c_n,a_1,\dots,a_k)$. Consider a valuation (for $Q_1^{(n+n_1)},\dots,Q_m^{(n+n_m)}$) in $\mathbb{F}$ such that $v \models Q_i(d_1,\dots,d_n,b_1,\dots,b_{n_i})$ iff

$$(v \models B_i(d_1,\dots,d_n,b_1,\dots,b_{n_i}) \text{ under the valuation } \xi)$$

for $1 \leq i \leq m$. Then $u \not\models A^{[n]}(c_1,\dots,c_n,a_1,\dots,a_k)$. ■

Note that Lemma 2 follows immediately from Lemma 6, since for any propositional formula A, $A^{[n]}$ is valid in $\mathbb{F}$ iff $A \in \mathbf{PL}(D_n,\leq_n)$.

A pre-ordered set $(W,\leq)$ is called *directed* if:

$$\forall u,v_1,v_2 \in W[(u \leq v_1, u \leq v_2) \Rightarrow \exists w \in W(v_1 \leq w, v_2 \leq w)].$$

It is well-known that $J \in \mathbf{PL}(W,\leq)$ iff $(W,\leq)$ is directed.

The following lemma is quite obvious:

LEMMA 7. *Let $(W,\leq)$ be a directed pre-ordered set. Let $u \leq w$, and w is a maximal element of W (i.e., $\forall v \geq w(v \leq w)$). Then $\forall v \geq u(v \leq w)$.*

3.1 Proof of Lemma 5, $(i) \Rightarrow (iii)$.

Assume that $\neg\neg F_\varepsilon^h \notin \mathbf{QL}(\mathbb{F})$. Take a valuation ξ in $\mathbb{F}, n \in \omega, u \in W, \mathbf{c} \in (D_u)^n$ such that $u \models \neg(F_\varepsilon^h)^{[n]}(\mathbf{c})$. Since $(D_n,\leq_n)$ is a pre-ordered set of a finite height ($\leq m$), we can assume that $\mathbf{c}$ is a $\leq_n$-maximal element of D_n (and thus u is maximal in W).

LEMMA 8. *For any $\mathbf{d} = (d_1,\dots,d_k) \in T_h^2, 0 \leq k \leq h$, there exist $u^{\mathbf{d}} \in W, (\mathbf{c}^{\mathbf{d}},\mathbf{a}^{\mathbf{d}}) \in D_{u^{\mathbf{d}}}^{(n+k^*)}$ such that:*

(a) $u \leq_0 u^{\mathbf{d}}, \mathbf{c} \leq_n \mathbf{c}^{\mathbf{d}}$;

(b) $u^{\mathbf{d}} \not\models R_{d_i}(\mathbf{c}^{\mathbf{d}},a_i^{\mathbf{d}}), u^{\mathbf{d}} \models R_{(1-d_i)}(\mathbf{c}^{\mathbf{d}},a_i^{\mathbf{d}})$ *for* $1 \leq i \leq k$
(here $\mathbf{a}^{\mathbf{d}} = (a_1^{\mathbf{d}},\dots,a_{k^*}^{\mathbf{d}})$ *);*

(c) $u^{\mathbf{d}} \not\models (R_0(\mathbf{c}^{\mathbf{d}},a_{k+1}^{\mathbf{d}}) \supset (F_{(\mathbf{d},1)}^h)^{[n]}(\mathbf{c}^{\mathbf{d}},\mathbf{a}^{\mathbf{d}}))\vee$
$(R_1(\mathbf{c}^{\mathbf{d}},a_{k+1}^{\mathbf{d}}) \supset (F_{(\mathbf{d},0)}^h)^{[n]}(\mathbf{c}^{\mathbf{d}},\mathbf{a}^{\mathbf{d}}))$ *for* $0 \leq k < h$ *;*

(d) $(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-}) \leq_{n+k} (\mathbf{c}^{\mathbf{d}}, (\mathbf{a}^{\mathbf{d}})^-)$,
where $\mathbf{d}^- = (d_1, \ldots, d_{k-1}), (\mathbf{a}^{\mathbf{d}})^- = (a_1^{\mathbf{d}}, \ldots, a_k^{\mathbf{d}})$, *for* $0 < k \leq h$.

Proof. By induction on k.
For $k = 0, \mathbf{d} = \varepsilon$ the claim holds, since

$$u \not\models \forall x_1[(R_0(\mathbf{c}, x_1) \supset (F^h_{(1)})^{[n]}(\mathbf{c}, x_1)) \vee (R_1(\mathbf{c}, x_1) \supset (F^h_{(0)})^{[n]}(\mathbf{c}, x_1))].$$

Let $k > 0$. By the inductive hypothesis,
$u^{\mathbf{d}^-} \not\models [R_{(1-d_k)}(\mathbf{c}^{\mathbf{d}^-}, a_k^{\mathbf{d}^-}) \supset (F^h_{\mathbf{d}})^{[n]}(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-})]$, where $\mathbf{d}^- = (d_1, \ldots, d_{k-1})$.
If $k < h$, we have:

$$(F^h_{\mathbf{d}})^{[n]}(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-}) = \forall x_{k+1}[(R_0(\mathbf{c}^{\mathbf{d}^-}, x_{k+1}) \supset (F^h_{(\mathbf{d},1)})^{[n]}(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-}, x_{k+1})) \vee \\ \vee (R_1(\mathbf{c}^{\mathbf{d}^-}, x_{k+1}) \supset (F^h_{(\mathbf{d},0)})^{[n]}(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-}, x_{k+1}))].$$

Hence there exists $(\mathbf{c}^{\mathbf{d}}, \mathbf{a}^{\mathbf{d}}) \in D^{n+k+1}$, satisfying the conditions $(c), (d)$, and $(b') : u^{\mathbf{d}} \models R_{(1-d_k)}(\mathbf{c}^{\mathbf{d}}, a_k^{\mathbf{d}})$. Also $u^{\mathbf{d}} \models R_{(1-d_i)}(\mathbf{c}^{\mathbf{d}}, a_i^{\mathbf{d}})$ for $i < k$, since $u^{\mathbf{d}^-} \models R_{(1-d_i)}(\mathbf{c}^{\mathbf{d}^-}, a_i^{\mathbf{d}^-}), (\mathbf{c}^{\mathbf{d}^-}, a_i^{\mathbf{d}^-}) \leq_{n+1} (\mathbf{c}^{\mathbf{d}}, a_i^{\mathbf{d}})$.

Finally, if $k = h$, then $(F^h_{\mathbf{d}})^{[n]}(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-}) = \bigvee\limits_{i=1}^{h} R_{d_i}(\mathbf{c}^{\mathbf{d}^-}, a_i^{\mathbf{d}^-})$.

Hence there exists $(\mathbf{c}^{\mathbf{d}}, \mathbf{a}^{\mathbf{d}})$ satisfying $(b), (d)$. Therefore, $u^{\mathbf{d}'} \not\models R_{d_i}(\mathbf{c}^{\mathbf{d}'}, a_i^{\mathbf{d}'})$ for any $i \leq j < h$ and $\mathbf{d}' = (d_1, \ldots, d_j)$, since $(\mathbf{c}^{\mathbf{d}'}, a_i^{\mathbf{d}'}) \leq_{n+1} (\mathbf{c}^{\mathbf{d}}, a_i^{\mathbf{d}})$. ■

The condition (ii) of Lemma 5 is obviously satisfied for our $(\mathbf{c}^{\mathbf{d}}, \mathbf{a}^{\mathbf{d}})$. Namely, $(\mathbf{c}^{\mathbf{d}}, a_i^{\mathbf{d}}) \not\leq_{n+1} (\mathbf{c}^{\mathbf{d}'}, a_j^{\mathbf{d}'})$ for any $\mathbf{d} = (d_1, \ldots, d_i, \ldots, d_k)$, $\mathbf{d}' = (d'_1, \ldots, d'_j, \ldots, d'_{k'})$ $(i \leq k \leq h, j \leq k' \leq h$) such that $d_i \neq d'_j$, i.e., $d'_j = 1 - d_i$, since $u^{\mathbf{d}} \models R_{d'_j}(\mathbf{c}^{\mathbf{d}}, a_i^{\mathbf{d}}), u^{\mathbf{d}'} \not\models R_{d'_j}(\mathbf{c}^{\mathbf{d}'}, a_j^{\mathbf{d}'})$. Also $\mathbf{c}^\varepsilon$ is maximal in D_n since $\mathbf{c} \leq_n \mathbf{c}^\varepsilon$. Without any loss of generality we can assume that $\mathbf{c} = \mathbf{c}^\varepsilon$ (and $u = u^\varepsilon$).

Let $(\mathbf{c}'^{\mathbf{d}}, \mathbf{e}^{\mathbf{d}})$ be a $\leq_{n+k+1}$-maximal element of D_{n+k+1} such that $(\mathbf{c}^{\mathbf{d}}, \mathbf{a}^{\mathbf{d}})$
$\leq_{n+k+1} (\mathbf{c}'^{\mathbf{d}}, \mathbf{e}^{\mathbf{d}})$, for $\mathbf{d} = (d_1, \ldots, d_k) \in T^2_{h-1}, 0 \leq k < h$ (recall that D_{n+k+1} is a pre-ordered set of finite height). Then

$$(\mathbf{c}^{\mathbf{d}'}, a_1^{\mathbf{d}'}, \ldots, a_{k+1}^{\mathbf{d}'}) \leq_{n+k+1} (\mathbf{c}'^{\mathbf{d}}, \mathbf{e}^{\mathbf{d}})$$

for any $\mathbf{d} = (d_1, \ldots, d_k), \mathbf{d}' = (d_1, \ldots, d_k, d'_{k+1}, \ldots, d'_h), \mathbf{a}^{\mathbf{d}'} = (a_1^{\mathbf{d}'}, \ldots, a_h^{\mathbf{d}'})$, by Lemma 7 (since $(\mathbf{c}^{\mathbf{d}}, \mathbf{a}^{\mathbf{d}}) \leq_{n+k+1} (\mathbf{c}^{\mathbf{d}'}, a_1^{\mathbf{d}'}, \ldots, a_{k+1}^{\mathbf{d}'})$).

LEMMA 9. *There exist* $b^{\mathbf{d}} \in D_u$ *(for* $\mathbf{d} \in T^2_{h-1}$*) such that* $(\mathbf{c}'^{\mathbf{d}}, \mathbf{e}^{\mathbf{d}}) \leq_{n+k+1}$ $\leq_{n+k+1} (\mathbf{c}, b^{\mathbf{d}_0}, \ldots b^{\mathbf{d}_k})$ *for any* $\mathbf{d} = (d_1, \ldots, d_k) \in T^2_{h-1}, 0 \leq k < h$ *(here* $\mathbf{d}_i = (d_1, \ldots, d_i)$ *for* $i \leq k$*, cf. Sect. 1.4).*

Proof. By induction on k (the length of $\mathbf{d}$).

For $k = 0, \mathbf{d} = \varepsilon$ we have $\mathbf{c}'^{\varepsilon} \leq_n \mathbf{c}$, since $\mathbf{c}$ is $\leq_n$-maximal in D_n; thus there exists $b_\varepsilon \in D_u$ such that $(\mathbf{c}'^{\varepsilon}, \mathbf{e}^{\varepsilon}) \leq_{n+1} (\mathbf{c}, b_\varepsilon)$ (by (π), Sect. 1.2).

Let $k > 0$. By the inductive hypothesis, $(\mathbf{c}'^{\mathbf{d}^-}, \mathbf{e}^{\mathbf{d}^-}) \leq_{n+k} (\mathbf{c}, b^{\mathbf{d}_0}, \ldots, b^{\mathbf{d}_{k-1}})$, where $\mathbf{d}^- = (d_1, \ldots, d_{k-1})$. On the other hand, $(\mathbf{c}'^{\mathbf{d}}, (\mathbf{e}^{\mathbf{d}})^-) \leq_{n+k} (\mathbf{c}'^{\mathbf{d}^-}, \mathbf{e}^{\mathbf{d}^-})$, where $\mathbf{e}^{\mathbf{d}} = (e_1^{\mathbf{d}}, \ldots e_k^{\mathbf{d}}, e_{k+1}^{\mathbf{d}}), (\mathbf{e}^{\mathbf{d}})^- = (e_1^{\mathbf{d}}, \ldots, e_k^{\mathbf{d}})$ (by Lemma 7, since $(\mathbf{c}^{\mathbf{d}^-}, \mathbf{a}^{\mathbf{d}^-}) \leq_{n+k} (\mathbf{c}^{\mathbf{d}}, (\mathbf{a}^{\mathbf{d}})^-) \leq_{n+k} (\mathbf{c}'^{\mathbf{d}}, (\mathbf{e}^{\mathbf{d}})^-)$). Hence, by (π), there exists $b^{\mathbf{d}} \in D_u$ such that $(\mathbf{c}'^{\mathbf{d}}, \mathbf{e}^{\mathbf{d}}) \leq_{n+k+1} (\mathbf{c}, b^{\mathbf{d}_0}, \ldots, b^{\mathbf{d}_{k-1}}, b^{\mathbf{d}})$. ■

The condition (iii) of Lemma 5 is satisfied for these $b^{\mathbf{d}} \in D_u$. Namely, $(\mathbf{c}^{\mathbf{d}'}, a_1^{\mathbf{d}'}, \ldots, a_{k+1}^{\mathbf{d}'}) \leq_{n+k+1} (\mathbf{c}'^{\mathbf{d}}, \mathbf{e}^{\mathbf{d}}) \leq_{n+k+1} (\mathbf{c}, b^{\mathbf{d}_0}, \ldots, b^{\mathbf{d}_k})$ for $\mathbf{d} = (d_1, \ldots, d_k), \mathbf{d}' = (d_1, \ldots, d_k, d'_{k+1}, \ldots, d'_h), 0 \leq k < h$.

Finally, assume that $(\mathbf{c}, b^{\mathbf{d}}) \leq_{n+1} (\mathbf{c}^{\mathbf{d}'}, a_i^{\mathbf{d}'})$ for some $\mathbf{d} = (d_1, \ldots, d_k)$, $\mathbf{d}' = (d'_1, \ldots, d'_i, \ldots, d'_j) (k < h, 1 \leq i \leq j \leq h)$.

Take $\mathbf{d}'' = (d_1, \ldots, d_k, 1 - d'_i, \ldots, d''_h)$. Then $(\mathbf{c}^{\mathbf{d}''}, a_{k+1}^{\mathbf{d}''}) \leq_{n+1} (\mathbf{c}, b^{\mathbf{d}}) \leq_{n+1}$ $\leq_{n+1} (\mathbf{c}^{\mathbf{d}'}, a_i^{\mathbf{d}'})$, which contradicts the condition $(ii)(2)$. ■

Note that if instead of finiteness of height (i.e., the validity of some P_m) we assume only that $\mathbf{PL}(\mathbb{F}) \not\subseteq \mathbf{PL}(W_2)$, then we have to construct $b^{\mathbf{d}}$ in a more accurate way. But this can be done, using the directedness of all D_k.

3.2 Proof of Lemma 5, $(ii) \Rightarrow (i)$.

Consider a valuation ξ in $\mathbb{F}$ such that (for $d \in \{0, 1\}$):

$$u \vDash R_d(\mathbf{c}, a) \text{ iff } [(\mathbf{c}^{\mathbf{d}}, a_k^{\mathbf{d}}) \leq_{n+1} (\mathbf{c}, a) \text{ for some } \\ \mathbf{d} = (d_1, \ldots, d_{k-1}, 1 - d) \in (T_2^h \backslash \{\varepsilon\})].$$

LEMMA 10. *For any* $\mathbf{d} = (d_1, \ldots, d_k) \in T_h^2$*:*

$$u^{\mathbf{d}} \nvDash (F_{\mathbf{d}}^h)^{[n]}(\mathbf{c}^{\mathbf{d}}, \mathbf{a}_k^{\mathbf{d}})$$

(here $\mathbf{a}_k^{\mathbf{d}} = (a_1^{\mathbf{d}}, \ldots, a_k^{\mathbf{d}})$ *if* $\mathbf{a}^{\mathbf{d}} = (a_1^{\mathbf{d}}, \ldots, a_{k^*}^{\mathbf{d}})$ *).*

Proof. By induction on $(h - k)$.

If $k = h$, we have $u^{\mathbf{d}} \nvDash R_{d_i}(\mathbf{c}^{\mathbf{d}}, a_i^{\mathbf{d}})$ for $i \leq h$, by the condition (2).

If $k < h$, we have

$$u^{\mathbf{d}} \nvDash \forall x_{k+1}[(R_0(\mathbf{c}^{\mathbf{d}}, x_{k+1}) \supset (F_{(\mathbf{d},1)}^h)^{[n]}(\mathbf{c}^{\mathbf{d}}, \mathbf{a}_k^{\mathbf{d}}, x_{k+1}))\vee \\ \vee(R_1(\mathbf{c}^{\mathbf{d}}, x_{k+1}) \supset (F_{(\mathbf{d},0)}^h)^{[n]}(\mathbf{c}^{\mathbf{d}}, \mathbf{a}_k^{\mathbf{d}}, x_{k+1}))],$$

since $u^{(\mathbf{d},d)} \vDash R_{1-d}(\mathbf{c}^{(\mathbf{d},d)}, a_{k+1}^{(\mathbf{d},d)}), u^{(\mathbf{d},d)} \nvDash (F_{(\mathbf{d},d)}^h)^{[n]}(\mathbf{c}^{(\mathbf{d},d)}, \mathbf{a}_k^{(\mathbf{d},d)}, a_{k+1}^{(\mathbf{d},d)})$, and $(\mathbf{c}^{\mathbf{d}}, \mathbf{a}^{\mathbf{d}}) = (\mathbf{c}^{\mathbf{d}}, \mathbf{a}_k^{\mathbf{d}}, a_{k+1}^{\mathbf{d}}) \leq_{n+k+1} (\mathbf{c}^{(\mathbf{d},d)}, \mathbf{a}_k^{(\mathbf{d},d)}, a_{k+1}^{(\mathbf{d},d)}) (= (\mathbf{c}^{(\mathbf{d},d)}, \mathbf{a}_{k+1}^{(\mathbf{d},d)}))$ for $d \in \{0, 1\}$, by the condition (1) (recall that $k^* = k+1$ for $k < h$). ■

Hence, for $\mathbf{d} = \varepsilon$ we conclude: $u^\varepsilon \not\models (F^h_\varepsilon)^{[n]}(\mathbf{c}^\varepsilon)$. Therefore, by the condition (0):

$$\forall u \geq u^\varepsilon \forall \mathbf{c} \in (D_u)^n[(\mathbf{c}^\varepsilon \leq_n \mathbf{c}) \Rightarrow u \not\models (F^h_\varepsilon)^{[n]}(\mathbf{c})].$$

Thus $u^\varepsilon \models (\neg F^h_\varepsilon)^{[n]}(\mathbf{c}^\varepsilon)$ and $u^\varepsilon \not\models (\neg\neg F^h_\varepsilon)^{[n]}(\mathbf{c}^\varepsilon)$. ∎

Acknowledgements

The author is grateful to anonymous referees who drew his attention to the paper by K. Nagaoka and E. Isoda.

The work on this paper was partly supported by Russian Foundation for Basic Research.

BIBLIOGRAPHY

[1] D. M. Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic*, forthcoming.

[2] S. Ghilardi. Presheaf semantics and independence results for some non-classical first-order logics. *Archive for Math. Logic*, **29**, 125–136, 1989.

[3] S. Ghilardi. Incompleteness results in Kripke semantics. *Journal of Symb. Logic*, **56**, 517–538, 1991.

[4] S. Ghilardi. Quantified extensions of canonical propositional intermediate logics. *Studia Logica*, **51**, 195-214, 1992.

[5] K. Nagaoka and E. Isoda. Incompleteness results in Kripke bundle semantics. *Math. Logic Quarterly*, **43**, 485-498, 1997.

[6] H. Ono. Model extension theorem and Craig's interpolation theorem for intermediate predicate logics. *Reports on Math. Logic*, **15**, 41–58, 1983.

[7] V. Shehtman and D. Skvortsov. Semantics of non-classical first order predicate logics. In *Mathematical Logic*, P. Petkov, ed., 105–116. Plenum Press, N.Y., 1990 (Proc. of Summer school and conference in math. logic Heyting'88).

[8] D. Skvortsov. An incompleteness result for intermediate predicate logics. *Journ. Symb. Logic*, **56**, 1145-1146, 1991.

All-Russian Institute of Scientific and Technical Information,
Molodogvardejskaja 22, korp.3, kv. 29
121351, Moscow
Russia
E-mail skvortsov@viniti.ru, skvortsov@lpcs.math.msu.ru