On IF Modal Logic and its Expressive Power

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ABSTRACT. *Independence-friendly (IF) modal logic* is defined and its expressive power is compared with that of basic modal logic. It is shown that for all $k \geq 1$, **IFML**$[k]$ (IF modal logic speaking of k accessibility relations) is relative to arbitrary k-ary modal structures (models with k accessibility relations) strictly more expressive than the corresponding basic modal logic **ML**$[k]$ (Theorem 5). An analogous result holds for the tense logics **TL**$[k]$ and **IFTL**$[k]$ (Theorem 7). On the other hand, over the class of unary temporal structures with linear frames, the tense logics **IFTL**$[1]$ and **TL**$[1]$ have the same expressive power (Theorem 9). Likewise the expressive powers of **IFML**$[1]$ and **ML**$[1]$ coincide on unary linear modal structures (Theorem 10). By contrast, for $k \geq n \geq 2$ and for k-ary models of whose accessibility relations n are linear, IF modal (*resp.* tense) logic is more expressive than the basic modal (*resp.* tense) logic (Theorem 11).

1 Introduction

My aim in this paper is to study whether allowing a modal operator to be logically independent of a specified subset of syntactically superordinate operators results in a modal language with extra expressive power as compared with basic modal logic.

First I recall the definitions of basic modal and basic tense logics of k accessibility relations (**ML**$[k]$ *resp.* **TL**$[k]$), and sketch their semantics in game-theoretical terms (*Section* 2). I then move on to obtain a language **IFML**$[k]$ from **ML**$[k]$ by allowing formulae with operator prefixes $O_1...O_{n-1}(O_n/W)$, where the intended interpretation of the indication "$/W$" is that the operator O_n is *logically (informationally) independent* of those preceding operators that are identified by indices in the set $W \subseteq \{1, ..., n-1\}$. This intuition is made precise by making use of games of imperfect information in defining the semantics for the language **IFML**$[k]$ (*Section* 3).

In *Section* 4 it is shown that the expressive power of **IFML**$[k]$ is strictly greater than that of **ML**$[k]$ relative to the class of all k-ary modal structures.

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A similar result about tense logics is proven for all k-ary temporal structures, whose accessibility relations are, as is usual, required to be in particular irreflexive and transitive. Further, it is shown that IF tense logic and basic tense logic evaluated relative to one *linear* accessibility relation have the same expressive power. From the proof of this result the analogous result follows for modal logics $\mathbf{IFML}[1]$ and $\mathbf{ML}[1]$. Finally, it is shown that these coincidence results are *not* generalizable to modal (temporal) structures with at least two linear accessibility relations.

2 Basic modal logic

We shall consider the two basic modalities for each *modality type* $i < \omega$: $\Diamond_i$, $\Box_i$. Occurrences of such operators are marked with unique indices (positive integers), as for instance in the string $\Diamond_{3,1}\Box_{1,2}\Box_{1,3}\Diamond_{2,4}p$. The formulae of *basic modal logic* $\mathbf{ML}[k]$ are generated from a fixed class **prop** of propositional atoms, and negations of propositional atoms, by the rules of closure under conjunction ($\wedge$), disjunction ($\vee$), and application of any of the (unary) modal operators $\Diamond_i$ and $\Box_i$ for $i < k$.

The semantics for $\mathbf{ML}[k]$ is defined in terms of *k-ary modal structures* $\mathcal{M} = (D, R_0, ..., R_{k-1}, \mathfrak{h})$, where D is a non-empty domain, the R_i are binary relations on D, and $\mathfrak{h}$ is a function assigning a subset of D to each propositional atom from **prop**. A pair $(\mathcal{M}, d)$ consisting of a modal structure and an element of its domain is called a *pointed modal structure*. The formulae of $\mathbf{ML}[k]$ are in effect evaluated relative to such pointed modal structures. We write $\mathcal{M} \models^+ \varphi[d]$ to make the judgement that the formula $\varphi \in \mathbf{ML}[k]$ is true in $\mathcal{M}$ at d; and we write $\mathcal{M} \models^- \varphi[d]$ to say that $\varphi \in \mathbf{ML}[k]$ is not true in $\mathcal{M}$ at d. The semantic clauses for formulae of $\mathbf{ML}[k]$ are completely standard:

- $\mathcal{M} \models^+ p[d] \iff d \in \mathfrak{h}(p)$
- $\mathcal{M} \models^+ \neg p[d] \iff d \notin \mathfrak{h}(p)$
- $\mathcal{M} \models^+ \varphi \vee \psi[d] \iff$ for some $\theta \in \{\varphi, \psi\}$: $\mathcal{M} \models^+ \theta[d]$
- $\mathcal{M} \models^+ \varphi \wedge \psi[d] \iff$ for every $\theta \in \{\varphi, \psi\}$: $\mathcal{M} \models^+ \theta[d]$

For each modality type $i < k$:

- $\mathcal{M} \models^+ \Diamond_i(\varphi)[d] \iff$ for some $c \in D$ with $R_i(d, c)$: $\mathcal{M} \models^+ \varphi[c]$.
- $\mathcal{M} \models^+ \Box_i(\varphi)[d] \iff$ for every $c \in D$ with $R_i(d, c)$: $\mathcal{M} \models^+ \varphi[c]$.

$\mathbf{ML}[0]$ evaluated over degenerate modal structures $(\{d_0\}, \mathfrak{h})$ is simply *Propositional logic*.

The semantics for a formula $\varphi \in \mathbf{ML}[k]$ can alternatively be given in terms of a semantical game $\mathrm{G}(\varphi, \mathcal{M}, d)$ between two players, *Abélard* ($\forall$) and *Héloïse* ($\exists$). A play of this game is defined in a straightforward fashion:

- If $\varphi \in \{p, \neg p\}$, no move is made. If $\varphi = p$ and $d \in \mathfrak{h}(p)$ or if $\varphi = \neg p$ and $d \notin \mathfrak{h}(p)$, then *Héloïse* wins the play and *Abélard* loses. Otherwise *Abélard* wins and *Héloïse* loses.
- If $\varphi = (\theta \wedge \psi)$ [*resp.* $\varphi = (\theta \vee \psi)$], then *Abélard* picks out a conjunct $\chi \in \{\theta, \psi\}$ [*resp.* *Héloïse* picks out a disjunct $\chi \in \{\theta, \psi\}$], and the play goes on as $\mathrm{G}(\chi, \mathcal{M}, d)$.
- Let $i < k$. If $\varphi = \Box_i(\psi)$ [*resp.* $\varphi = \Diamond_i(\psi)$], then *Abélard* [*resp.* *Héloïse*] picks out, if possible, a state d' with $R_i(d, d')$; and the play continues as $G(\psi, \mathcal{M}, d')$. If however such a choice is not possible (i.e. if d is R_i-maximal), then *Héloïse* [*resp.* *Abélard*] wins and *Abélard* [*resp.* *Héloïse*] loses.

A *strategy* of the player $j \in \{\forall, \exists\}$ in the game $\mathrm{G}(\varphi, \mathcal{M}, d)$ is any function providing a move for this player j in any situation where it is according to the rules of the game j's turn to move. A strategy is *winning* (in short, a *w.s.*) for the player whose strategy it is, if using this strategy the player wins any play of the game $\mathrm{G}(\varphi, \mathcal{M}, d)$ - irrespectively, that is, of the moves of his or her opponent.

Truth and falsity for $\mathbf{ML}[k]$-formulae are then defined game-theoretically as follows:

- $\mathcal{M} \models^{+}_{\mathrm{GTS}} \varphi[d] \iff$ there exists a w.s. for *Héloïse* in $\mathrm{G}(\varphi, \mathcal{M}, d)$.
- $\mathcal{M} \models^{-}_{\mathrm{GTS}} \varphi[d] \iff$ there exists a w.s. for *Abélard* in $\mathrm{G}(\varphi, \mathcal{M}, d)$.

It is easy to check that the above recursively defined semantics to $\mathbf{ML}[k]$ and the game-theoretical semantics coincide:[1]

$$\mathcal{M} \models^{+} \varphi[d] \iff \mathcal{M} \models^{+}_{\mathrm{GTS}} \varphi[d]$$

$$\mathcal{M} \models^{-} \varphi[d] \iff \mathcal{M} \models^{-}_{\mathrm{GTS}} \varphi[d]$$

The semantics to $\mathbf{ML}[k]$ could well be defined more generally than is done above, by allowing evaluation relative to *tuples* of elements from a specified Cartesian product $D_0 \times ... \times D_{N_k} - 1$ with $1 \leq N_k \leq k$. Then for each (binary) accessibility relation R_i a component $D_{\pi(i)}$ of the Cartesian product would

[1] In general, the directions from left (the recursive definition) to right (the GTS-definition) require *Axiom of Choice*.

have to be associated by some surjection $\pi : \{0, ..., k-1\} \to \{0, ..., N_k - 1\}$. With an eye to many applications such a more flexible semantics would be useful. If for instance we'd wish to consider two modality types such as physical necessity and temporality, we would probably find it desirable to have one co-ordinate for possible worlds and another for time in our semantics. On the other hand, for instance in multiagent epistemic logic, k equals the number of the agents while the parameter N_k has the value 1, since formulae of that logic are evaluated relative to single possible worlds.

The semantics of *basic tense logic* $\mathbf{TL}[k] = \mathbf{ML}[2k]$ employs $2k$-ary modal structures

$$\mathcal{M} = (D, R_0, ..., R_{2k-1}, \mathfrak{h})$$

with the characteristic features that: **(i)** for each $j < k$, the relation R_{j+k} is the *converse* of the relation R_j; **(ii)** the accessibility relations are *irreflexive partial orders*, in other words irreflexive and transitive binary relations. The motivation for the restriction **(i)** is that the inverse of an operator O (which by definition always exists in a tense logic) must make use of the converse of precisely the accessibility relation associated with O. The feature **(ii)** is required, because at least irreflexivity and transitivity are thought of as essential to the temporal *earlier than* -relation. In connection with $\mathbf{TL}[k] = \mathbf{ML}[2k]$, we shall by convention write $\Diamond_i^{-1}$ *resp.* $\Box_i^{-1}$ for the operators $\Diamond_{i+k}$ *resp.* $\Box_{i+k}$ $(i < k)$.

$2k$-ary modal structures $\mathcal{M}$ satisfying the above conditions **(i)** and **(ii)** are termed k-ary *temporal* structures. By contrast, we shall call $2k$-ary modal structures meeting the requirement **(i)** but not the requirement **(ii)** *quasi-temporal* structures.

3 IF modal logic and its uniformity interpretation

I proceed to define syntax and semantics to an extension of basic modal logic I call *IF modal logic of* k *modality types*, or $\mathbf{IFML}[k]$. Thereby more dependencies and independencies among modal operators will become expressible than are expressible within $\mathbf{ML}[k]$.

3.1 The language

The language of $\mathbf{IFML}[k]$ is simply this:

$$\mathbf{IFML}[k] := \mathbf{ML}[k] \cup \{O_1...O_{n-1}(O_n/W)\varphi : \varphi \in \mathbf{ML}[k], n \geq 1\},$$

where:

- for all $j \in \{1, ..., n\}$, O_j is one of the modal operators $\Diamond_{i,j}$, $\Box_{i,j}$ for some $i < k$.

- W is a (possibly empty) subset of the interval $[1, n-1]$ of natural numbers.

The intuitive reading of the expression $(\Diamond_{i,n}/W)$ is "there is a state s_n such that $R_i(s_{n-1}, s_n)$, *independently* of the states s_j with $j \in W$". The intuitive reading of $(\Box_{i,n}/W)$ is analogous.

The class of well-formed formulae for IF modal logic could well be defined more generally. In principle for every modal operator (and even for every propositional connective), its own independence indication "$/W$" could be added, where W would be an (empty or nonempty) set of indices of other operators or connectives appearing in the same formula. In the present paper, however, we stick to the definition of $\mathbf{IFML}[k]$ given above, as already this class turns out to be interesting as compared with the basic modal logic $\mathbf{ML}[k]$.

IF *tense* logic of k temporal modality types, or $\mathbf{IFTL}[k]$, is defined to be the logic $\mathbf{IFML}[2k]$ with the same stipulations about notation as were introduced above in connection with basic tense logic $\mathbf{TL}[k] = \mathbf{ML}[2k]$.

3.2 Semantics intuitively

The semantics of $\mathbf{IFML}[k]$ is going to be given game-theoretically. For each formula $\varphi \in \mathbf{IFML}[k]$, a semantical game is defined [relative to a k-ary pointed modal structure $(\mathcal{M},d)$] by extending the above game-theoretical definition of the semantics for $\mathbf{ML}[k]$ to cover also formulae of the form $O_1...O_{n-1}(O_n/W)\varphi$. The key semantic idea in dealing with these new formulae is to make use of the game-theoretical notion of *informational independence* in defining how to evaluate the operator (O_n/W). Roughly, it will be required that a winning strategy of a player be *uniform* in those components of its arguments that correspond to operators O_j with $j \in W$, i.e. that distinct moves made for these operators O_j - of which the operator O_n is declared to be *independent* - in different possible plays of the semantical game never suffice by themselves for prompting a distinct move for O_n. Whichever move is made for O_n after *one* distribution of moves corresponding to the operators O_j with $j \in W$, is made - other things being equal, i.e. as long as the choices for the operators with $j \notin W$ are the same - for *all* distributions that can come up in the course of a play of the game. Different plays corresponding to evaluating the block $O_1...O_{n-1}$ are partitioned into equivalence classes so that plays h and h' are in the same class if and only if they differ at most for moves corresponding to operators with indices from W (that is, agree at least for indices from $\{1, ..., n-1\} \backslash W$); and the relevant player's strategy can only be winning if it respects these information sets in the sense that it gives histories from the same the set a common value.

We now move on to define the semantics in a precise way. We need to have the notion of substring of an $\mathbf{IFML}[k]$-formula available. The class $Sub(\varphi)$ of *proper substrings* of an $\mathbf{IFML}[k]$-formula is defined recursively for strings φ *including* all $\mathbf{IFML}[k]$-formulae - but *not all* of which are formulae of this logic. (We cannot simply define here the class of sub*formulae* of $\mathbf{IFML}[k]$, since substrings $O_i...O_{n-1}(O_n/W)\varphi$ - which we wish to have always in the class $Sub(\varphi)$ - are by the syntax *not* formulae of $\mathbf{IFML}[k]$, if W is not contained in $\{i, ..., n-1\}$.) The clauses for (negated) propositional atoms and Boolean connectives are standard: for an atom $p \in \mathbf{prop}$, $Sub(p) = Sub(\neg p)$ is empty; and $Sub(\varphi \wedge \psi) = Sub(\varphi \vee \psi) = \{\varphi, \psi\} \cup Sub(\varphi) \cup Sub(\psi)$. If $W \subseteq \{1, ..., n-1\}$ and $1 \leq i \leq n$, we define $Sub(O_i...O_{n-1}(O_n/W)\varphi) = \{O_{i+1}...O_{n-1}(O_n/W)\varphi\} \cup Sub(O_{i+1}...O_{n-1}(O_n/W)\varphi)$. Finally, by stipulation the string `FAIL` is in $Sub(\varphi)$ for all $\varphi \in \mathbf{IFML}[k]$. The strings in the class $Sub(\varphi) \cap \mathbf{IFML}[k]$ are said to be *subformulae* of φ.

3.3 The semantical games

A semantical game $\mathrm{G}_A(\varphi, \mathcal{M}, d) =$

$$\langle \{\forall, \exists\}, H, Z, P, \{u_\forall, u_\exists\}, \{I_\forall, I_\exists\} \rangle$$

in extensive normal form is associated with each triple $(\varphi, \mathcal{M}, d)$ consisting of a formula φ of $\mathbf{IFML}[k]$; a k-ary modal structure $\mathcal{M} = (D, R_0, ..., R_{k-1}, \mathfrak{h})$; and an element $d \in D$. *Positions* in this game are pairs (ψ, a) from the set

$$A = (Sub(\varphi) \cup \{\varphi\}) \times (dom(\mathcal{M}) \cup \{\star\}).$$

The object $\star$ goes together with the substring `FAIL`; the pair (`FAIL`,$\star$) is a position that can be chosen in situations where there were otherwise no choice complying with the game rules available at all. The game $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ is a game between two players, *Héloïse* (or $\exists$) and *Abélard* (or $\forall$). The players are said to be *opponents* of each other. The set H of *plays* (or, *histories*) of $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ is a set of *finite sequences of positions* (ψ, a), defined recursively (on the subformula structure of φ) as follows:[2]

- $(\varphi, d) \in H$.
- If the last position in $h \in H$ is $(\theta \vee \psi, a)$, then $h^\frown(\theta, a) \in H$ and $h^\frown(\psi, a) \in H$.
  It is *Héloïse* who makes a choice from $\{(\theta, a), (\psi, a)\}$ to extend h. We write $P(h) = \exists$ to indicate that the move corresponding to the history h is made by $\exists$.

[2] If $h = (\alpha_1, ..., \alpha_m)$ and $\alpha \in A$, we write "$h^\frown\alpha$" (read: h *extended* by α) to denote the sequence $(\alpha_1, ..., \alpha_m, \alpha)$.

- The case where the last position of $h \in H$ is $(\theta \wedge \psi, a)$ is defined analogously, the only difference being that here it is *Abélard* who makes the choice: $P(h) = \forall$.

For $i < k$:

- If the last position in $h \in H$ is from $\{((\Diamond_{i,n}/W)\psi, a), (\Diamond_{i,n}\psi, a)\}$, then for all $a' \in dom(\mathcal{M})$ with $R_i(a, a')$, $h^\frown(\psi, a') \in H$;
  if no such $a' \in dom(\mathcal{M})$ exists, then $h^\frown(\texttt{FAIL}, \star) \in H$. Further, $P(h) = \exists$.

- The definitions for $\Box_{i,n}/W$ and $\Box_{i,n}$ are like those for $\Diamond_{i,n}/W$ and $\Diamond_{i,n}$, except that here it is $\forall$'s turn to move.

All sequences in the set H are finite. The set Z of *terminal histories* is defined simply as the set of maximally long histories, i.e. sequences from H which cannot be extended by any position so as to yield a sequence in H. By definition, then, the last position in a terminal history has as its substring-component a propositional atom, the negation of a propositional atom, or the label FAIL. The function $P : H \backslash Z \to \{\forall, \exists\}$ constructed above simultaneously with the definition of H will be called the *player function*.

The *utility functions* $u_\forall$ and $u_\exists$ for the two players of the game are maps from the class Z of terminal histories to the values 1 (a win) and -1 (a loss) as follows. For any $h \in Z$, $u_\exists(h) = 1$ and $u_\forall(h) = -1$, if the last position of h is of the form (p, a) and $a \in \mathfrak{h}(p)$, or is of the form $(\neg p, a)$ and $a \notin \mathfrak{h}(p)$, or is the position $(\texttt{FAIL}, \star)$ chosen by $\forall$. Otherwise $u_\forall(h) = 1$ and $u_\exists(h) = -1$.

The *information partitions* $I_\forall$ and $I_\exists$ for the two players still need be defined. For an arbitrary history $h = ((\varphi_0, a_0), ..., (\varphi_m, a_m))$, we call the sequence $pr_1(h) = (\varphi_0, ..., \varphi_m)$ its *left projection*; and the sequence $pr_2(h) = (a_0, ..., a_m)$ its *right projection*. If $(s_0, ..., s_n)$ is any finite sequence and $i \leq n$, we write $(s_0, ..., s_n)[i]$ for its member s_i. Now the sets $P^{-1}(\{\exists\})$ and $P^{-1}(\{\forall\})$ are partitioned into equivalence classes under the following equivalence relations $\sim_\exists$ and $\sim_\forall$, respectively.

- $h_1 \sim_\exists h_2 \iff$
  (for some $a_1, a_2 \in dom(\mathcal{M})$, the last position in h_1 is $((\Diamond_{i,n}/W)\theta, a_1)$ and the last position in h_2 is $((\Diamond_{i,n}/W)\theta, a_2)$; and $pr_1(h_1) = pr_1(h_2)$; and for all $j \notin W$, $pr_2(h_1)[j] = pr_2(h_2)[j]$) **or**
  ($h_1 = h_2$ and its last position is of the form (ψ, a) for some $\psi \in \{\Diamond_{i,n}\theta, (\theta \vee \chi)\}$ and $a \in dom(\mathcal{M})$).

The condition under which $h_1 \sim_\forall h_2$ holds is similar to the condition for $\sim_\exists$ and is obtained from it by replacing the occurrences of the symbols

"$\Diamond$" and "$\vee$" by the symbols "$\Box$" and "$\wedge$", respectively. The promised partitions of $P^{-1}(\{\exists\})$ and $P^{-1}(\{\forall\})$ then are:

$$I_\exists = \{[h]_{\sim_\exists} : h \in H \backslash Z\} \text{ and } I_\forall = \{[h]_{\sim_\forall} : h \in H \backslash Z\}.$$

The sets $I_\exists$ and $I_\forall$ themselves are called *information partitions*, and their members (i.e. the cells of these partitions) are called *information sets*.

The definition of the semantical game

$$\mathrm{G}_A(\varphi, \mathcal{M}, d) = \langle \{\forall, \exists\}, H, Z, P, \{u_\forall, u_\exists\}, \{I_\forall, I_\exists\} \rangle$$

in its extensive form has become completed. By definition it is a zero-sum game. And it is a game of *imperfect* information.

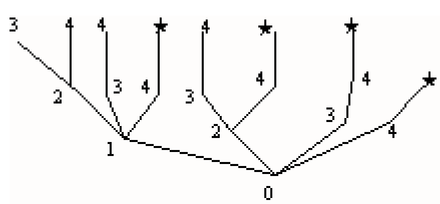
3.4 Imperfect information modelling logical independence

Intuitively the idea with information partitions is that histories in the same cell of the information partition of the player $j \in \{\exists, \forall\}$ are indistinguishable to this player: unless there is some way of ruling out a subset of such an information set as histories that definitely cannot have been reached in the course of a play of the game, any move j makes after one history belonging to this set, j must be able to make after *any* history from this set, in order for j's strategy for making these moves to be winning: j's winning strategy has to agree on all these histories. To get a better grasp of these partitions, let us take an example.

Example Let $\mathcal{M} = (\{0, 1, 2, 3, 4\}, <, \mathfrak{h})$ be a modal structure, where $<$ is the ordering of the natural numbers 0,1,2,3,4 by magnitude, the assignment $\mathfrak{h}$ being arbitrary. Consider the formulae:

$$(\mathbf{a}) \quad \varphi := \Box_1 \Diamond_2 \Diamond_3 / \{1\} p.$$
$$(\mathbf{b}) \quad \psi := \Box_1 \Diamond_2 \Diamond_3 / \{2\} p.$$

Up to indicating the relevant information partitions, the extensive form of both semantical games, $\mathrm{G}_A(\varphi, \mathcal{M}, 0)$ and $\mathrm{G}_A(\psi, \mathcal{M}, 0)$, is depicted as below.



Let the histories of length 2 that appear in the picture be listed as $h_1, ..., h_7$ from left to right. Now in the case of the formula **(a)**, the information partition for *Héloïse* contains, in addition to four singletons each consisting of a history of length 1, the following cells whose members are histories of length 2:

$$\{h_1\}, \{h_2, h_4\}, \{h_3, h_5, h_6\} \text{ and } \{h_7\}.$$

In the same cell are those histories of length 2 that have their last member in common. By contrast, in the case of the formula (**b**), the cells of *Héloïse*'s information partition consisting of histories of length 2 are these:

$$\{h_1, h_2, h_3\}, \{h_4, h_5\}, \{h_6\} \text{ and } \{h_7\}.$$

Here the histories in the same cell share their second member (i.e. the first member chosen after the initial position $(\psi, 0)$).

3.5 Uniformity and plans of action

To make use of the games $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ in defining semantics to $\mathbf{IFML}[k]$, we need the notion of strategy of a player. A *strategy* of the player $j \in \{\exists, \forall\}$ is simply any function

$$f_j : P^{-1}(\{j\}) \to A.$$

Observe that strategies are *not* components in the extensive form of a semantical game. Given that $h \in H$ is a history on which a strategy f_j is defined, we say that the move $f_j(h)$ given by the strategy is *legal*, if $h^\frown f_j(h) \in H$, i.e. when the extension to h determined by f_j remains inside the set H of histories of the relevant game. Otherwise the move is said to be *illegal*.

We shall employ the following terminology. If s, s_1 and s_2 are sequences such that $s = s_1{}^\frown s_2$, then s_1 is said to be an *initial segment* of s. If s_2 is non-empty, s_1 is a *proper* initial segment of s. When S is a set of sequences, we write $Cl(S)$ for the set of initial segments of members of S, i.e. for the closure of S under forming initial segments.

A strategy f_j of the player j is *winning* (is a *w.s.*), if there exists a subset $W \subseteq Z$ of terminal histories satisfying the following four conditions:

- **(a)** If $h \in Cl(W)$ and $P(h)$ is j, then $h^\frown f_j(h) \in Cl(W)$.
- **(b)** If $h \in Cl(W)$ and $P(h)$ is the opponent of j, then for every $u \in A$ such that $h^\frown u \in H$, $h^\frown u \in Cl(W)$.
- **(c)** For every $h, h' \in Cl(W)$: if $h, h' \in \mathrm{I} \in I_j$, then $f_j(h) = f_j(h')$.
- **(d)** Every $h \in W$ is a win for j.

The condition **(a)** simply says that $Cl(W)$ is closed under applications of the strategy f_j; and **(c)** expresses the 'uniformity condition' for this strategy. The condition **(b)** in meant to ensure that j wins against any move of his opponent. Together these conditions are meant to ensure that only the plays that are **"**reached**"** given that j uses the strategy f_j are relevant for its being winning: the only plays in which the strategy f_j ever needs be used, are those from $Cl(W)$.

A set $W \subseteq Z$ which thus establishes that a strategy is winning, is called a *plan of action*. We shall say that f_j is a winning strategy *based on* W. We observe that W guarantees the existence of a minimal (though not necessarily unique) partial function from $P^{-1}(\{j\})$ to A which specifies a move for all combinatorially possible moves of the opponent of j and always yields a win for j.

If there exists a set $W \subseteq Z$ satisfying the conditions **(a)**, **(b)** and **(c)** but not necessarily **(d)**, we say that a corresponding *strategy* f_j is *based on the plan of action* W. But unless the condition **(d)** is satisfied, such a strategy is *not* winning.

Fact 1 *At most one of the players has a w.s. in a game* $\Gamma = \mathrm{G}_A(\varphi, M, d)$.

Proof. Assume for contradiction that strategies $f_\exists$ and $f_\forall$ are both winning in Γ. Fix W *resp.* W' as corresponding plans of action. We show that there is a history h such that $h \in W$ and $h \in W'$, which yields a contradiction by the condition **(d)**.

We construct h in steps. For the initial position $\alpha_1 = (\varphi, d)$, we trivially have that $(\alpha_1) \in Cl(W)$ and $(\alpha_1) \in Cl(W')$. Assume then that we have reached $(\alpha_1, ..., \alpha_m)$ such that $(\alpha_1, ..., \alpha_m) \in Cl(W)$ and $(\alpha_1, ..., \alpha_m) \in Cl(W')$. If $P((\alpha_1, ..., \alpha_m)) = \forall$, then by **(a)**, $(\alpha_1, ..., \alpha_m, f_\forall(\alpha_1, ..., \alpha_m)) \in Cl(W')$. But then by the clause **(b)** $(\alpha_1, ..., \alpha_m, f_\forall(\alpha_1, ..., \alpha_m)) \in Cl(W)$. The same reasoning applies if $P((\alpha_1, ..., \alpha_m)) = \exists$. Hence in a finite number of steps a terminal history $h = (\alpha_1, ..., \alpha_N)$ is reached that is both in W and in W'; and so by **(d)** h is a win for both players, which is impossible. ■

For instance Ariel Rubinstein ([9], p. 66) suggests that it is natural to define strategies in terms of plans of action — i.e. as functions which essentially specify the actions of a player only for such histories that he or she may actually confront when applying this plan. This is in contrast to the typical game-theoretical practice of requiring that strategies be defined for all histories of the game, irrespective of whether such a history can be reached when following that strategy.

3.6 Semantics to IFML

Now that the plethora of requisite definitions all are given, we can state the semantics for $\mathbf{IFML}[k]$. Let $\mathcal{M} = (D, R_0, ..., R_{k-1}, \mathfrak{h})$ be any k-ary modal structure; and let $d \in D$. Then:

- $\varphi \in \mathbf{IFML}[k]$ is true in $\mathcal{M}$ at d, in symbols $\mathcal{M} \models^+ \varphi[d]$, iff there exists a w.s. for *Héloïse* in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.
- $\varphi \in \mathbf{IFML}[k]$ is false in $\mathcal{M}$ at d, in symbols $\mathcal{M} \models^- \varphi[d]$, iff there exists a w.s. for *Abélard* in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$.
- $\varphi \in \mathbf{IFML}[k]$ is non-determined in $\mathcal{M}$ at d, in symbols $\mathcal{M} \models^0 \varphi[d]$, iff $\mathcal{M} \not\models^+ \varphi[d]$ and $\mathcal{M} \not\models^- \varphi[d]$.

The semantics just given to $\mathbf{IFML}[k]$ will be referred to as its *uniformity interpretation*.

It is clear that for the subclass $\mathbf{ML}[k]$ of $\mathbf{IFML}[k]$, the game-theoretical semantics defined in *Section* 2 coincides with the above game-theoretical semantics.

For $k = 0$, $\mathbf{IFML}[k]$ and $\mathbf{ML}[k]$ are the same languages. Hence, $\mathbf{IFML}[0]$ = $\mathbf{ML}[0]$ evaluated over degenerate modal structures $(\{d_0\}, \mathfrak{h})$ is indeed *Propositional logic*. For $k \geq 1$, $\mathbf{ML}[k]$ is a proper subclass of $\mathbf{IFML}[k]$.

The formulae of IF *tense logic* $\mathbf{IFTL}[k] = \mathbf{IFML}[2k]$ are evaluated relative to k-ary temporal structures. (Recall that k-ary temporal structures are by definition a subclass of all $2k$-ary modal structures.)

4 Comparing the Expressive Powers of ML and IFML

If $\mathcal{L}$ and $\mathcal{L}'$ are modal logics (IF or not) and $\mathcal{C}$ is a class of modal structures on which the semantics of these logics is defined, it is said that $\mathcal{L}$ is *embeddable in $\mathcal{L}'$ over $\mathcal{C}$* (symbolically $\mathcal{L} \leq_{\mathcal{C}} \mathcal{L}'$), if for each formula $\varphi \in \mathcal{L}$ there exists a formula $\psi_\varphi \in \mathcal{L}'$ such that for all $\mathcal{M} \in \mathcal{C}$ and all $d \in dom(\mathcal{M})$,

$$\varphi \text{ is true in } \mathcal{M} \textit{ at } d \iff \psi_\varphi \text{ is true in } \mathcal{M} \textit{ at } d.$$

If $\mathcal{L}$ is embeddable in $\mathcal{L}'$ over $\mathcal{C}$ but not *vice versa*, it is said that $\mathcal{L}'$ has *greater expressive power* than $\mathcal{L}$ over $\mathcal{C}$ (in symbols $\mathcal{L} <_{\mathcal{C}} \mathcal{L}'$). If $\mathcal{L}$ is embeddable in $\mathcal{L}'$ and *vice versa*, then it is said that they have *the same expressive power* over $\mathcal{C}$ (symbolically $\mathcal{L} =_{\mathcal{C}} \mathcal{L}'$).

4.1 Basic facts about expressive power

Let us write $\mathcal{C}_k$ for the class of all k-ary modal structures. For an arbitrary $k < \omega$, we have completely trivially the following embeddability result:

Fact 2 *Over C_k, $\mathbf{ML}[k]$ is embeddable in $\mathbf{IFML}[k]$.*

This is just because $\mathbf{ML}[k]$ is a subclass of $\mathbf{IFML}[k]$: any formula $\varphi \in \mathbf{ML}[k]$ is its own translation in $\mathbf{IFML}[k]$.

We say that the class $\mathbf{IFML}_{det}[k] :=$

$$\{\varphi \in \mathbf{IFML}[k] : for\ all\ \mathcal{M} \in \mathcal{C}_k\ and\ all\ d \in dom(\mathcal{M}),$$

$$\mathcal{M} \models^+ \varphi[d]\ or\ \mathcal{M} \models^- \varphi[d]\}$$

is the *determined fragment* of the logic $\mathbf{IFML}[k]$. It is straightforward to show that relative to the class $\mathcal{C}_k$ of all k-ary modal structures, the expressive powers of $\mathbf{IFML}_{det}[k]$ and $\mathbf{ML}[k]$ are the same:

THEOREM 1. $\mathbf{IFML}_{det}[k] =_{\mathcal{C}_k} \mathbf{ML}[k]$.

Proof. All formulae of $\mathbf{ML}[k]$ are determined (everywhere either true or false), and they are also (directly by syntactical criteria) formulae of $\mathbf{IFML}[k]$, so $\mathbf{ML}[k] \leq_{\mathcal{C}_k} \mathbf{IFML}_{det}[k]$. For the other direction, let $\varphi \in \mathbf{IFML}_{det}[k]$, $\mathcal{M} \in \mathcal{C}_k$ and $d \in dom(\mathcal{M})$ all be arbitrary. We have to find a formula $\psi_\varphi \in \mathbf{ML}[k]$, satisfying that

$$\text{there is a } w.s. \text{ for } \exists \text{ in } \mathrm{G}_A(\varphi, \mathcal{M}, d) \Longleftrightarrow \mathcal{M} \models^+_{\mathrm{GTS}} \psi_\varphi\,[d]\,.$$

Now if in particular $\varphi \in \mathbf{ML}[k]$, then we may clearly take ψ_φ to be φ itself. If, again, φ is of the form $O_1...O_{n-1}(O_n/W)\chi$ with $\chi \in \mathbf{ML}[k]$, then put $\psi_\varphi := O_1...O_{n-1}O_n\chi$. What can be done in greater ignorance can be done in lesser ignorance: clearly if some $f_\exists$ is a w.s. for $\exists$ in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$, the same strategy is a w.s. for her in $\mathrm{G}_A(\psi_\varphi, \mathcal{M}, d)$. Assume then that there is *no* w.s. for $\exists$ in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$. But then, by the assumption of determinacy, there is a w.s. for $\forall$ in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$. And a w.s. for $\forall$ in $\mathrm{G}_A(\varphi, \mathcal{M}, d)$ is clearly also a w.s. for him in $\mathrm{G}_A(\psi_\varphi, \mathcal{M}, d)$. This means, in turn, that there is no w.s. for $\exists$ in $\mathrm{G}_A(\psi_\varphi, \mathcal{M}, d)$. ■

We shall need subsequently the notion of *bisimulation* and the simple *Invariance Lemma* relating the relation of bisimilarity to the capacity of $\mathbf{ML}[k]$-formulae to distinguish k-ary modal structures.

DEFINITION 2. (*Bisimulation*) Let $\mathcal{M} = (D, R_0, ..., R_{k-1}, \mathfrak{h})$ and $\mathcal{N} = (D', R'_0, ..., R'_{k-1}, \mathfrak{h}')$ be k-ary modal structures. A *bisimulation* between the modal structures $\mathcal{M}$ and $\mathcal{N}$ is a binary relation $\equiv_{D,D'} \subseteq D \times D'$, satisfying the following three conditions **(1)**, **(2)** and **(3)**.

(1) *Atomic Harmony:*
$d \equiv_{D,D'} d' \Longrightarrow$ for all $p \in \mathbf{prop}$: $d \in \mathfrak{h}(p) \Longleftrightarrow d' \in \mathfrak{h}'(p)$.

(2) *Zigzag Forwards:* for all $i < k$,
$d \equiv_{D,D'} d'$ and $R_i(d,c) \Longrightarrow \exists c'(R'_i(d',c')$ and $c \equiv_{D,D'} c')$.

(3) *Zigzag Backwards:* for all $i < k$,
$d \equiv_{D,D'} d'$ and $R'_i(d',c') \Longrightarrow \exists c(R_i(d,c)$ and $c \equiv_{D,D'} c')$.

A *bisimilarity* $B_{D,D'}$ between modal structures $\mathcal{M}$ and $\mathcal{N}$ is the *union* of all bisimulations between $\mathcal{M}$ and $\mathcal{N}$. The pointed modal structures $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are said to be *bisimilar*, if there is a bisimilarity $B_{D,D'}$ such that $(d, d') \in B_{D,D'}$. Further, we say that the pointed modal structures $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are $\mathbf{ML}[k]$-*equivalent*, if for all $\varphi \in \mathbf{ML}[k]$,

$$\mathcal{M} \models^+ \varphi[d] \Longleftrightarrow \mathcal{N} \models^+ \varphi[d'].$$

The notion of $\mathbf{IFML}[k]$-equivalence of $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ is defined similarly.

For formulae of $\mathbf{ML}[k]$ the following *Invariance Lemma* holds. The lemma says that bisimilarity preserves truth in $\mathbf{ML}[k]$ in the sense that if $(\mathcal{M}, d)$ and $(\mathcal{N}, d')$ are bisimilar, then these pointed modal structures are $\mathbf{ML}[k]$-equivalent.

LEMMA 3. *(Invariance Lemma) Let $\mathcal{M} = (D, R_0, ..., R_{k-1}, \mathfrak{h})$ and $\mathcal{N} = (D', R'_0, ..., R'_{k-1}, \mathfrak{h}')$ be k-ary modal structures. Then for all $\varphi \in \mathbf{ML}[k]$, all $d \in D$ and all $d' \in D'$:*

$$(d, d') \in B_{D,D'} \Longrightarrow (\mathcal{M} \models^+ \varphi[d] \Longleftrightarrow \mathcal{N} \models^+ \varphi[d']).$$

Proof. A straightforward induction on the complexity of the formula $\varphi \in \mathbf{ML}[k]$. ■

4.2 Expressive power over arbitrary modal structures

Let us next consider the modal logics $\mathbf{ML}[1]$ and $\mathbf{IFML}[1]$, which hence both are evaluated relative to precisely one accessibility relation. Let $\mathcal{C}_1$ be the class of all unary modal structures. We prove that $\mathbf{IFML}[k]$ has a greater expressive power than $\mathbf{ML}[k]$ relative to $\mathcal{C}_1$:

LEMMA 4. $\mathbf{ML}[1] <_{\mathcal{C}_1} \mathbf{IFML}[1]$.

Proof. Construct unary modal structures $\mathcal{M}$ and $\mathcal{M}'$ as follows.
Put $\mathcal{M} = (D, R, \mathfrak{h})$ and $\mathcal{M}' = (D', R', \mathfrak{h}')$, where
- $D = D' = \{a, b, c, d, e, f\}$ (6 distinct elements)
- $R = R' = \{(a, b), (a, c), (b, d), (b, e), (c, e), (c, f)\}$

• $\mathfrak{h}(q) = \{e\}$, $\mathfrak{h}'(q) = \{d, f\}$
Define $\equiv\, \in Pow(D \times D')$ to be the following relation:

$$\{(a,a),(b,b),(c,c),(b,c),(c,b)(d,e),(f,e)(e,d),(e,f)\}.$$

It is easy to check that $\equiv$ is a bisimulation between the unary modal structures $\mathcal{M}$ and $\mathcal{M}'$. Hence, by *Invariance Lemma*, the pointed modal structures $(\mathcal{M}, a)$ and $(\mathcal{M}', a)$ are **ML**[1]-equivalent. On the other hand, we observe that the **IFML**[1]-formula $\Box_{0,1}\Diamond_{0,2}/\{1\}q$ is true in $\mathcal{M}$ at a, but is non-determined in $\mathcal{M}'$ at a. It follows that **IFML**[1] is not embeddable in **ML**[1]. By appealing to Fact 2 the proof is completed. ■

Notice that the frames (D, R) and (D', R') of the modal structures employed in the proof of Lemma 4 are isomorphic, in fact identical.[3] Hence, by the proof, **IFML**[1] is able to distinguish properties properly of models (as opposed to properties of frames) which are not distinguishable using **ML**[1].

Remark. The result, provided by Lemma 4, that **IFML**[1] has greater expressive power than **ML**[1] over the class $\mathcal{C}_1$ of all unary modal structures, is *not* a corollary to the completely trivial fact that there exist **IFML**[1]-formulae (such as the formula $\Box_{0,1}\Diamond_{0,2}/\{1\}q$ employed in the proof of Lemma 4) that are non-determined in certain pointed modal structures. Obviously for such a formula φ there cannot exist an **ML**[1]-formula ψ_φ sharing the truth-value with φ in all structures $(\mathcal{M}, d)$ with $\mathcal{M} \in \mathcal{C}_1$, $d \in dom(\mathcal{M})$ - this is because all **ML**[1]-formulae are everywhere determined but φ by definition isn't. Had we defined embeddability of a logic $\mathcal{L}$ to a logic $\mathcal{L}'$ over a class $\mathcal{C}$ by the condition that for each $\varphi \in \mathcal{L}$ there must exist $\psi \in \mathcal{L}'$ such that for all $\mathcal{M} \in \mathcal{C}$ and for all $d \in dom(\mathcal{M})$:

φ and ψ are both true, both false, or both non-determined in $\mathcal{M}$ at d

and not by the condition we actually used (i.e. that for all $\varphi \in \mathcal{L}$ there is $\psi \in \mathcal{L}'$ that is *true* in precisely the same structures $(\mathcal{M}, d)$ in which φ is *true*), this trivial fact would indeed suffice for showing that **IFML**[1] is strictly more expressive than **ML**[1] over $\mathcal{C}_1$. But it does *not* follow from the fact that the truth-value of a given non-determined $\varphi \in$ **IFML**[1] never coincides in all structures $(\mathcal{M}, d)$ with the truth-value of any $\psi_\varphi \in$ **ML**[1] that there could not exist a formula $\psi_\varphi \in$ **ML**[1] such that φ and ψ_φ have all their *verifying* models in common - it being an additional question what

[3]Frames $(D, R_0, ..., R_{k-1})$ and $(D', R'_0, ..., R'_{k-1})$ are said to be *isomorphic* when there exists a bijection $f : D \to D'$ such that for all $d_1, d_2 \in D$ and for all $i < k$, $R_i(d_1, d_2) \iff R'_i(f(d_1), f(d_2))$.

would happen with those structures in which $\varphi \in \mathbf{IFML}[1]$ were not true (i.e. were false or non-determined). Hence - our definition of embeddability being what it is - the sole existence of non-determined $\mathbf{IFML}[1]$-formulae is by itself an insufficient ground for inferring Lemma 4.

Lemma 4 has the straightforward consequence that for any positive $k < \omega$, $\mathbf{IFML}[k]$ is strictly more expressive than $\mathbf{ML}[k]$ over the class $\mathcal{C}_k$ of arbitrary k-ary modal structures:

THEOREM 5. *For any positive $k < \omega$, $\mathbf{ML}[k] <_{\mathcal{C}_k} \mathbf{IFML}[k]$.*

By Theorem 5, we have for all positive $k < \omega$ that $\mathbf{TL}[k] <_{\mathcal{C}_{2k}} \mathbf{IFTL}[k]$. By the proof of Lemma 4, this result can be improved to hold for the proper subclass QT_k of $\mathcal{C}_{2k}$ consisting of all k-ary *quasi-temporal* structures:

COROLLARY 6. *For any positive $k < \omega$, $\mathbf{TL}[k] <_{QT_k} \mathbf{IFTL}[k]$.*

Proof. The statement of the corollary will trivially follow, if this statement is established for the value $k = 1$. To do so, let $\mathcal{M} = (M, R, \mathfrak{h})$ and $\mathcal{M}' = (M', R', \mathfrak{h}')$ be the unary modal structures defined in the proof of Lemma 4 above. Then the structures $\mathcal{M}^+ = (M, R, R^{-1}, \mathfrak{h})$ and $\mathcal{M}'^+ = (M', R', R'^{-1}, \mathfrak{h}')$, obtained from $\mathcal{M}$ and $\mathcal{M}'$ by adding to these the converses of their accessibility relations, are in the class QT_1. But in fact the relation $\equiv$ defined in the proof of Lemma 4 is a bisimulation between the binary modal structures $\mathcal{M}^+$ and $\mathcal{M}'^+$. And the formula $\Box_{0,1}\Diamond_{0,2}/\{1\}q$ is true in $\mathcal{M}^+$ at a, but non-determined in $\mathcal{M}'^+$ at a. Hence the statement follows. ∎

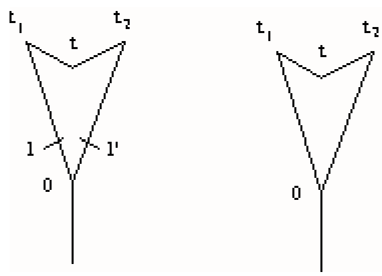
However, the structures $\mathcal{M}^+$ and $\mathcal{M}'^+$ appearing in the proof of Corollary 6 are *not* genuine temporal structures - in particular the relations R and R' are not transitive. But it is possible to prove a genuine tense logical analogue to Theorem 5. Furthermore, the proof can be given so that the structures witnessing the result have isomorphic frames. Let us write $\mathcal{T}_k$ for the class of all k-ary temporal structures. (Recall that these are quasi-temporal strctures whose all accessibility relations are irreflexive partial orders.)

THEOREM 7. *For any positive $k < \omega$, $\mathbf{TL}[k] <_{\mathcal{T}_k} \mathbf{IFTL}[k]$.*

Proof. Let $(\mathrm{C}_1,<_1)$ be the set of positive rationals ordered by magnitude. Write $\mathrm{C}_2 := \mathrm{C}_1 \times \{0\}$, and let $(\mathrm{C}_2,<_2)$ be an isomorphic copy of $(\mathrm{C}_1,<_1)$. Finally, let t, t_1, t_2 be three *irrational* numbers. Define a frame (D, R) as follows:

- $D := \mathbb{Q} \cup C_2 \cup \{t, t_1, t_2\}$;
- $R := < \cup [\mathbb{Q} \times \{t_1\}] \cup [(\mathbb{Q} \backslash C_1) \times (C_2 \cup \{t_2\})] \cup <_2 \cup$

$[C_2 \times \{t_2\}] \cup \{(t,t_1),(t,t_2)\}$;
where $\mathbb{Q}$ stands for the set of all rational numbers, and $<$ for their order by magnitude. Finally, let **prop** $= \{q\}$, and define assignments $\mathfrak{h}$ and $\mathfrak{h}'$ by:
• $\mathfrak{h}(q) = \{1, 1'\}$; $\mathfrak{h}'(q) = \{0\}$;
where $1'$ is the image of 1 under a fixed isomorphism that establishes the isomorphism between $(C_1, <_1)$ and $(C_2, <_2)$. The relation R is clearly irreflexive and transitive. The picture below illustrates the temporal structures $\mathcal{M} = (D, R, R^{-1}, \mathfrak{h})$ and $\mathcal{N} = (D, R, R^{-1}, \mathfrak{h}')$.



The structures $\mathcal{M}$ and $\mathcal{N}$ have a *common* frame (D, R, R^{-1}); and so *a fortiori* their frames are isomorphic. Furthermore, the temporal structures in question are bisimilar, as is witnessed by the relation

$$\{(t,t)\} \cup [\{t_1,t_2\} \times \{t_1,t_2\}] \ \cup \ \{(1,0),(1',0)\} \cup$$
$$[\{x : 1 <_1 x <_1 t_1 \textbf{ or } 1' <_2 x <_2 t_2\} \times \{x : 0 <_1 x <_1 t_1 \textbf{ or } 0 <_2 x <_2 t_2\}] \ \cup$$
$$[\{x : x <_1 1 \textbf{ or } x <_2 1'\} \times \{x : x <_1 0\}].$$

Hence the pointed temporal structures $(\mathcal{M}, t)$ and $(\mathcal{N}, t)$ cannot be distinguished by any **TL**[1]-formula. On the other hand, the **IFTL**[1]-formula

$$\varphi := \Box_1 \Diamond_2^{-1}/\{1\}q$$

is non-determined in $\mathcal{M}$ at t; but it is true in $\mathcal{N}$ at t. Hence we may conclude that **IFTL**[1] is more expressive than **TL**[1] over the class of all unary temporal structures. The statement of the theorem then trivially follows. ■

4.3 Expressive power and linear frames

We shall say that a k-ary modal structure $\mathcal{M}$ is *linear in n dimensions*, if $\mathcal{M}$ has $n \leq k$ accessibility relations which are irreflexive linear orders. A k-ary *temporal* structure is said to be linear in n dimensions, if it has $2n \leq 2k$ accessibility relations which are irreflexive linear orders. We adopt the convention of writing **LO**$[k, n]$ for the class of k-ary modal structures linear in n dimensions. By stipulation we write $\mathbf{LO}[k] := \mathbf{LO}[k,k]$. The class of k-ary temporal structures linear in n dimensions we denote by writing $\mathbf{LO}_{temp}[k,n]$; and we stipulate that $\mathbf{LO}_{temp}[k] := \mathbf{LO}_{temp}[k,k]$. In what follows we always assume that the linear orders considered are *irreflexive*.

It can be shown that over unary temporal structures $(D, R, R^{-1}, \mathfrak{h})$ with an arbitrary irreflexive linear order R, the expressive powers of the tense logics $\mathbf{TL} := \mathbf{TL}[1]$ and $\mathbf{IFTL} := \mathbf{IFTL}[1]$ coincide. The following lemma is useful in the proof. (It holds for *all* logics $\mathbf{IFML}[k]$ and *all* k-ary modal structures.)

LEMMA 8. *Let* $W \subseteq [1, n-1]$*; and let* $U = [K+1, n-1] \subseteq W$*, where* $K \notin W$ $(0 \leq K \leq n-1)$*. For all* k*-ary modal structures* $\mathcal{M}$*; all* $d \in dom(\mathcal{M})$*; and all* $\varphi \in$ ***IFML****[k], we have:*

Héloïse has a w.s. in $G(O_1 ... O_{n-1}(O_n/W)\varphi, \mathcal{M}, d) \Longleftrightarrow$
Héloïse has a w.s. in $G(O_1 ... O_{n-1}(O_n/U)\varphi, \mathcal{M}, d)$.

Proof. The set $W \subseteq [1, n-1]$ clearly has a unique presentation as a disjoint union $W = V' \cup [K+1, n-1]$, where $K \notin W$. The implication from left to right is trivial. (What can be done in greater ignorance can be done in lesser ignorance.) Assume then that there is a w.s. (f) for *Héloïse* in $\Gamma = G_A(O_1 ... O_{n-1}(O_n/U)\varphi, \mathcal{M}, d)$. We observe that for any $i \in \{1, ..., n-1\}$, if *Héloïse* knows the state s_i chosen for the operator O_i (which is the case exactly when $i \notin W$), then in order to evaluate O_n, the choices made to evaluate operators with indices $j < i$ are of no relevance. The 'route' by which the play of the game has proceeded to the choice of this state s_i is irrelevant. And K precisely is *not* contained in W, nor in $U = [K+1, n-1]$. This is why *Héloïse*'s w.s. (f^*) in $\Gamma^* = G_A(O_1 ... O_{n-1}(O_n/W)\varphi, \mathcal{M}, d)$ can be constructed using her w.s. (f) in Γ in such a way that for each cell I of the information partition corresponding to O_n in Γ^* we choose some such subset I' of I which is a cell of the information partition corresponding to O_n in Γ (in general *Axiom of Choice* is needed for choosing I'); and then we define f^* on I by the value f gives to the histories from I'. By the observation just made, this value really works also for all other cells of Γ contained in the cell I of Γ^*, and thereby legally extends all histories from I. ■

THEOREM 9. $\mathbf{TL} =_{\mathbf{LO}_{temp}[1]} \mathbf{IFTL}$.

Proof. $\mathbf{TL}$ is trivially embeddable in $\mathbf{IFTL}$ over any class of structures. For the other direction, consider $\mathbf{IFTL}$-formulae of the form $\psi = O_1 ... O_{n-1}$ $(O_n/W)\varphi$. By Lemma 8 the set W referred to in ψ can be assumed to be of the form $[K+1, n-1]$. Indeed, we may suppose that $W = \{1, ..., n-1\}$, since clearly

$$O_1 ... O_K O_{K+1} ... O_{n-1}(O_n/\{K+1, ..., n-1\})\varphi \Longleftrightarrow O_1 ... O_K \chi,$$

if χ is a **TL**-translation of the **IFTL**-formula $O_{K+1}...O_{n-1}(O_n/\{K+1, ..., n-1\})\varphi$. The cases $W = \varnothing$ and $O_n \in \{\Box, \Box^{-1}\}$ are trivial. The laborious part of the proof is to show that the **IFTL**-formulae of the form $O_1...O_{n-1}(O_n/\{1, ..., n-1\})\varphi$ (where $O_n \in \{\Diamond, \Diamond^{-1}\}$, $n \geq 2$) can be translated to **TL**. By symmetry it suffices to consider formulae of the form $O_1...O_{n-1}(\Diamond_n/\{1, ..., n-1\})\varphi$. On their part the proof is by dividing the proof into subcases according to the distribution of operators in the prefix $O_1...O_{n-1}$ of the formula ψ, and taking into account different properties the linear order in the model may have. The relevant subcases are:

- **Case 1.** For all $i \in \{1, ..., n-1\}$: $O_i \in \{\Diamond_i\}$.
- **Case 2.** For all $i \in \{1, ..., n-1\}$: $O_i \in \{\Diamond_i, \Box_i^{-1}\}$; and $\Box^{-1}$ appears at least once in the block.
- **Case 3.** There is last occurrence of $\Diamond^{-1}$ in the block, and after it all operators are from $\{\Diamond, \Box^{-1}\}$.
- **Case 4.** There is last occurrence of $\Box$ in the block, and after it all operators are from $\{\Diamond, \Box^{-1}\}$.

Case 1 is trivial and cases 3 and 4 can be handled by reducing them to case 2. The case 2 is proven by showing that the (negation normal form of the) formula $\theta_\psi :=$

$$O_1...O_{n-1}\Diamond_n\varphi \wedge \neg\Diamond^{-1}(\neg\varphi \wedge \Box\neg\varphi \wedge \Box^{-1}\Diamond\varphi) \wedge \neg(\neg\varphi \wedge \Box\neg\varphi \wedge \Box^{-1}\Diamond\varphi)$$

counts as a required **TL**-translation of ψ. (The string θ_ψ is not in **TL**, as it contains $\neg$ prefixed to non-atomic formulae; it is a formula of the logic obtained from **TL** by closing **TL** under negation, and it has a unique negation normal form which is in **TL**.) The idea with the two latter conjuncts in θ_ψ is that they ban the possibility that the first conjunct $(O_1...O_{n-1}\Diamond_n\varphi)$ would be true at t while there was a point $s \leq t$ such that

- φ is false at s and at all points later than s; but
- for all $s' < s$ there is s'' such that $s' < s'' < s$ and φ is true at s''.

The properties of the order that need be taken into account when proving the four cases are whether or not the point of evaluation is discrete to the left *resp.* to the right, and if yes, how many discrete moves can be made in the respective direction; and also whether the domain has or does not have a minimum *resp.* a maximum relative to its linear order. The relevant different cases concerning the order of course prompt fresh disjuncts to the translation in question. ■

The proof of Theorem 9, when spelled out in detail, shows that the expressive powers of **ML**[1] and **IFML**[1] coincide on unary modal structures with an arbitrary linear order:

THEOREM 10. $\mathbf{ML}[1] =_{\mathbf{LO}[1]} \mathbf{IFML}[1]$.

Let us finally consider binary modal structures linear in both dimensions. We may observe that on the one hand, Theorem 9 shows that **IFML**[2] does *not* have more expressive power than **ML**[2] over the *subclass* $\mathbf{LO}_{temp}[1]$ of **LO**[2] consisting of *such* binary modal structures linear in both dimensions whose two accessibility relations are *converses to each other* (i.e. over unary linear *temporal* structures). On the other hand, by Theorem 5 we know that **IFML**[2] *has* a greater expressive power than **ML**[2] over *arbitrary* binary modal structures. The case we now move on to consider is hence an intermediate one: since

$$\mathbf{LO}_{temp}[1] \subset \mathbf{LO}[2] \subset \mathcal{C}_2$$

is a chain of proper inclusions, the theorems 5 and 9 do not suffice for deciding the question whether the relation $\mathbf{ML}[2] <_{\mathbf{LO}[2]} \mathbf{IFML}[2]$ holds or not. The following theorem implies that in fact **IFML**[2] is already over **LO**[2] more expressive than **ML**[2]. More generally, the theorem states that the equalities $\mathbf{ML}[k] =_{\mathbf{LO}[k,n]} \mathbf{IFML}[k]$ and $\mathbf{TL}[k] =_{\mathbf{LO}_{temp}[k,n]} \mathbf{IFTL}[k]$ which by the theorems 9 and 10 are established for the value $k = n = 1$, cannot be generalized to any values $k \geq n \geq 2$.

THEOREM 11. *For all $n \geq 2$ and for all $k \geq n$, we have:*
(a) $\mathbf{ML}[k] <_{\mathbf{LO}[k,n]} \mathbf{IFML}[k]$.
(b) $\mathbf{TL}[k] <_{\mathbf{LO}_{temp}[k,n]} \mathbf{IFTL}[k]$.

Proof. We shall show that each of the claims **(a)** and **(b)** holds for the values $k := 2$ and $n := 2$. The results then trivially generalize to all $k \geq n \geq 2$. Now let $\mathcal{M} = (D, <_0, <_1, \mathfrak{h})$ and $\mathcal{M}' = (D', <'_0, <'_1, \mathfrak{h}')$ be binary modal structures whose components are given as follows.

- D is the set of rational numbers, and $<_0 = <_1$ is the order of rationals by magnitude.

Let $C = D \times \{0\}$, and let $(C, <_0^+, <_1^+)$ be an isomorphic copy of $(D, <_0, <_1)$. By construction $C \cap D = \varnothing$. Then put:

- $D' := D \cup C$
- $<'_0 := <_0 \cup <_0^+ \cup (D \times C)$
- $<'_1 := <_1 \cup <_1^+ \cup (C \times D)$

Finally, assume for simplicity that **prop** $= \{\top\}$ and put $\mathfrak{h}(\top) = D$ and $\mathfrak{h}'(\top) = D'$. The structures $\mathcal{M}$ and $\mathcal{M}'$ can respectively be depicted as below.

It is clear that the relation $D \times D'$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Fix then points $\zeta \in D$ and $\xi \in D' \cap C$. It follows by *Invariance Lemma* that the pointed modal structures $(\mathcal{M}, \zeta)$ and $(\mathcal{M}', \xi)$ cannot be distinguished by any formula of **ML**[2]. But we observe that the **IFML**[2]-formula $\Box_{0,1}\Diamond_{1,2}/\{1\}\top$ is true in $\mathcal{M}'$ at ξ (*Héloïse* can choose for $\Diamond_{1,2}$ any $\xi'' \in D' \cap D$, as any such ξ'' will be a $<_1'$-successor of any possible ξ' with $\xi <_0' \xi'$ chosen by *Abélard*); but on the other hand this formula is non-determined in $\mathcal{M}$ at ζ (not true, since no $\zeta'' \in D$ is a $<_1$-successor to all $\zeta' \in D$ with $\zeta <_0 \zeta'$; not false, as for all $\zeta' \in D$ there is some $\zeta'' \in D$ such that $\zeta <_0 \zeta' <_1 \zeta''$). It follows that **IFML**[2] is not embeddable in **ML**[2]. By appealing to Fact 2, the claim **(a)** is hence shown for the value $k := 2$; and the generalized claim **(a)** can be inferred.

Then observe that as both $<_0$ and $<_1$ are irreflexive and transitive relations, the structures

$$\mathcal{M}^+ = (D, <_0, <_1, <_0^{-1}, <_1^{-1}, \mathfrak{h}) \text{ and } \mathcal{M}'^+ = (D', <_0', <_1', <_0'^{-1}, <_1'^{-1}, \mathfrak{h}')$$

are temporal structures. But the relation $D \times D'$ is a bisimulation even between these binary temporal structures $\mathcal{M}^+$ and $\mathcal{M}'^+$. Hence $(\mathcal{M}^+, \zeta)$ and $(\mathcal{M}'^+, \xi)$ are **TL**[2]-equivalent but can of course be distinguished by the formula $\Box_{0,1}\Diamond_{1,2}/\{1\}\top$. Hence the claim **(b)** follows. ■

Summary of the relations between the logics considered

The relative expressive powers of the logics considered in this paper are seen to be as follows.

Arbitrary modal structures:

$$\begin{aligned} &\mathbf{ML}\,[0] =_{\mathcal{C}_0} \mathbf{IFML}\,[0] <_{\mathcal{C}_1} \mathbf{ML}\,[1] <_{\mathcal{C}_1} \mathbf{IFML}\,[1] \ldots \\ &\ldots <_{\mathcal{C}_k} \mathbf{ML}\,[k] <_{\mathcal{C}_k} \mathbf{IFML}\,[k] \ldots \end{aligned}$$

After the first members $\mathbf{ML}[0] =_{\mathcal{C}_0} \mathbf{IFML}[0]$, this is a hierarchy of ever increasing expressive power. However, once the relation $\mathbf{ML}\,[1] <_{\mathcal{C}_1} \mathbf{IFML}\,[1]$ is shown to hold, the fact that the other relations hold follows trivially.

Arbitrary temporal structures:

$$\begin{aligned}&\mathbf{TL}\,[0] =_{\mathcal{T}_0} \mathbf{IFTL}\,[0] <_{\mathcal{T}_1} \mathbf{TL}\,[1] <_{\mathcal{T}_1} \mathbf{IFTL}\,[1] \ldots\\ &<_{\mathcal{T}_k} \mathbf{TL}\,[k] <_{\mathcal{T}_k} \mathbf{IFTL}\,[k] \ldots\end{aligned}$$

This hierarchy of tense logics is completely analogous to the above hierarchy among modal logics; and holds relative to genuine temporal structures.

By the definitional identities $\mathbf{TL}[k] = \mathbf{ML}[2k]$ and $\mathbf{IFTL}[k] = \mathbf{IFML}[2k]$ (and the transitivity of the relation "of greater expressive power than"), the above hierarchy of modal logics includes a hierarchy of tense logics evaluated relative to *modal* structures. The above tense logical hierarchy is relative to *temporal* structures, and thereby it is an improvement of the result already known from the modal logical case.

Modal logics and linear structures:

$$\begin{aligned}&\mathbf{ML}\,[1] =_{\mathbf{LO}[1]} \mathbf{IFML}\,[1] <_{\mathbf{LO}[2,n_2]} \mathbf{ML}\,[2] <_{\mathbf{LO}[2,n_2]} \mathbf{IFML}\,[2] \ldots\\ &\ldots <_{\mathbf{LO}[k,n_k]} \mathbf{ML}\,[k] <_{\mathbf{LO}[k,n_k]} \mathbf{IFML}\,[k] \ldots,\end{aligned}$$

where for all $i \geq 2, n_i \geq 2$.

Tense logics and linear structures:

$$\begin{aligned}&\mathbf{TL}\,[1] =_{\mathbf{LO}_{temp}[1]} \mathbf{IFTL}\,[1] <_{\mathbf{LO}_{temp}[2,n_2]} \mathbf{TL}\,[2] <_{\mathbf{LO}_{temp}[2,n_2]}\\ &\mathbf{IFTL}\,[2] \ldots <_{\mathbf{LO}_{temp}[k,n_k]} \mathbf{TL}\,[k] <_{\mathbf{LO}_{temp}[k,n_k]} \mathbf{IFTL}\,[k] \ldots,\end{aligned}$$

where for all $i \geq 2, n_i \geq 2$.

5 Related work

The basic framework of this paper stems from the independence-friendly (IF) first-order logic of Hintikka and Sandu [5, 10], which allows presenting arbitrary logical priorities among logical expressions (quantifiers and propositional connectives) in first-order formulae. Henkin quantifier formulae are examples of formulae expressible in IF first-order logic but not in the usual first-order logic (**FO**). (For Henkin quantifiers, see [3, 4, 7, 12].) They are however by no means the only representatives for patterns of logical expressions that cannot be equivalently expressed in usual first-order logic. Other examples are provided by the partially ordered connectives (conjunctions, disjunctions) studied by Sandu and Väänänen [11]; and mutually dependent quantifiers discussed recently on various occasions by Hintikka (see e.g. [6]).

The systematical motivation for the present paper has been the following: as it is possible to define a first-order logic capable of exhibiting more freedom (in fact, full freedom) in relations of logical priority among its logical expressions than usual first-order logic can (a fact witnessed by the existence of IF first-order logic), we may ask whether the same could be done in the case of modal logic. Is there a reasonable way to define 'IF modal logic' in which scope relations among modal operators are relaxed as compared with 'basic modal logic', in a way analogous to the way in which relations of priority scope are relaxed among quantifiers in IF first-order logic? And if so, what can be said about the relative expressive powers of such an IF modal logic and its traditional sibling?

The possibility of defining IF modal logic has been studied in Bradfield [1] and Bradfield & Fröschle [2]. The way IF modal logic is formulated in these papers results in a very expressive logic. In fact Bradfield's general framework makes his IF modal logic more expressive than the logics I have studied in the present paper: his 'Henkin-modalities' are not expressible in the logics studied here. (It can be shown that **IFML**$[k]$ can be translated into usual first-order logic.) The motivation of the studies by Bradfield and Fröschle has been to investigate whether the logical notion of independence appearing in IF logics could prove useful for *concurrency theory* in computer science. My approach in defining IF modal logic in the present paper proceeds from the traditional modal logical setting in a more straightforward way than does Bradfield's approach. In both cases informational independence is made use of when defining the semantics. Bradfield, however, wishes to further require that those transitions of which a given transition is to be independent, must all be concurrent with that independent transition.

Thinking of further conceivable applications to IF modal logic, linguistic theorizing and in particular the analysis of tenses might be a case in point. There is a tradition stemming from Prior [8] of construing natural language tenses as sentential operators. Most contemporary linguists (e.g. Enç, Hornstein, Kamp) oppose, in one way or another, to construing tenses as operators. On the other hand, all that would be needed for an ineliminable theoretical role for operators in the case of tenses is that it be possible that the interpretation of one tense be logically (functionally) dependent on the interpretation of another tense, in analogy to the way in which the interpretation of the quantifier $\exists y$ is dependent on the interpretation of the quantifier $\forall x$ in a formula of the form $\forall x \exists y\ \varphi$. If no such functional dependencies ever occur with tenses, then their sole role is to introduce times and they may as well be taken for pronominal expressions, or for adverbs (which act only very locally). But if such dependency can occur, then there are instances of natural language tenses as operators. When suitably elaborated, IF tense logic offers a natural framework for discussing the logical role of tenses in natural language as opposed to other features connected with tense semantics, notably the mechanism of choosing the temporal interpretation of a tense as temporally related (via a relation such as *earlier than* or *contemporaneousness*) to the interpretation of a syntactically superordinate tense.

6 Conclusion

We have shown that modifying basic modal logic by dissociating the syntactical substring-relation and the relation of logical dependence results in a modal logic whose expressive power is greater than that of basic modal logic over arbitrary modal structures (Theorem 5); an analogous fact holds true of corresponding tense logics and genuine temporal structures (Theorem 7). A case especially worthy of notice is the case of unary *linear* structures. The expressive power of IF tense logic does *not* add to that of basic tense logic in the case of unary linear temporal structures (Theorem 9). Similarly, the expressive powers of IF modal logic and basic modal logic coincide over unary linear modal structures (Theorem 10). It was proven that adding more linear accessibility relations renders the resulting IF modal (tense) logic again more expressive than its basic sibling (Theorem 11).

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